Value distribution of Ramanujan sums and of cyclotomic polynomial coefficients

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Abstract

The Ramanujan sum $c_n(k)$ and $a_n(k)$, the kth coefficient of the nth cyclotomic polynomial, are completely symmetric expressions in terms of primitive nth roots of unity. For k fixed we study the value distribution of $c_n(k)$ (following A. Wintner) and $a_n(k)$ (partly following H. Möller). In particular we disprove a 1970 conjecture of H. Möller on the average (over n) of $a_n(k)$. We show that certain symmetric functions in primitive roots considered by the Dence brothers are related to the behaviour of $c_{p-1}(k)$ and $a_{p-1}(k)$ as p ranges over the primes and study their value distribution.

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1 Introduction

We recall from elementary number theory the notion of multiplicative order. If a and n are coprime natural numbers, then there is a smallest positive integer k such that $a^k \equiv 1(\text{mod } n)$, this integer k is the *multiplicative order* of a, $\text{ord}_n(a)$, and divides $\varphi(n)$, where φ is Euler's totient (discussed more extensively in the next section). We say that a is a *primitive root* of the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^*$ of units of $\mathbb{Z}/n\mathbb{Z}$ if $\text{ord}_n(a) = \varphi(n)$. This means that $(\mathbb{Z}/n\mathbb{Z})^*$ is cyclic and a is a generator. Let p be a prime (indeed, throughout we exclusively use the notation p and q for primes). It is well-known that $(\mathbb{Z}/p\mathbb{Z})^*$ is cyclic and hence it has at least one primitive root. Indeed, if g is a primitive root mod p, then so are g^j with j and $p-1$ coprime and hence there are $\varphi(p-1)$ distinct primitive roots mod p. Let us denote them by $g_1, ..., g_t$ with $t = \varphi(p-1)$ and $0 < g_i \leq p-1$. In their paper [5] the Dence brothers consider symmetric functions of the primitive roots of primes. They consider for example

$$s_2(p) = \sum_{1 \leq i < j \leq t} g_i g_j \tag{1}$$

and show that this quantity, when considered mod p, assumes only the values in $\{-1, 0, 1\}$. They write in particular (p. 79): 'As a matter of distribution, we observe that amongst the first 100 primes (beginning with $p = 5$) the residues $-1, 0, 1$ of s_2 occur in the ratios $12 : 59 : 29$. The question of what these ratios should be in the limit of infinitely many primes is an interesting one'. The following table gives the ratios amongst the first 10^j primes with $2 \leq j \leq 6$ (thus beginning with $p = 2$).

Table 1: Value distribution of $s_2(p)$

$\pi(x)$	$s_2(p) = -1$	$s_2(p) = 0$	$s_2(p) = 1$
10^2	0.110000	0.610000	0.280000
10^3	0.099000	0.625000	0.276000
10^4	0.093000	0.626100	0.280900
10^5	0.094120	0.627330	0.278550
10^6	0.093939	0.626216	0.279845

The sum in (1) is the second elementary totally symmetric function of $g_1, \cdots, g_t$. For $k \geq 1$ let us consider more generally $s_k(p)$ with $s_k(p)$ the kth order totally elementary symmetric function in $g_1, \cdots, g_t$.

Question 1. Fix $k \geq 1$. As p ranges over the primes, which values are assumed by $s_k(p)(\text{mod } p)$ and with what frequency ?

Likewise we consider, for $k \geq 1$, the sum

$$S_k(p) = \sum_{1 \leq j \leq t} g_j^k.$$

For this quantity we consider the same question:

Question 2. Fix $k \geq 1$. As p ranges over the primes, which values are assumed by $S_k(p) (\bmod p)$ and with what frequency ?

One of the motivations of this M.Sc. thesis is to resolve these questions. Interestingly, we will see that symmetric functions in primitive roots are closely related with some important objects in number theory; cyclotomic polynomials and Ramanujan sums. Both cyclotomic polynomials and Ramanujan sums arise in many contexts and hence we devote separate sections to them.

We will see that the question of the Dence brothers naturally leads to the study of the value distribution of Ramanujan sums and of that of coefficients of cyclotomic polynomials. These two issues have not been well studied in the literature. Regarding the value distribution of Ramanujan sums we could only find an almost forgotten paper by the famous analyst Aurel Wintner [25], which we reconsider in §7. Regarding the value distribution of coefficients of cycltomic polynomials, we reconsider a paper by Herbert Möller [12] in §8 and establish some new results.

2 Preliminaries

2.1 Multiplicative functions

An *arithmetic function* is a function from the natural numbers to the complex numbers. A very important subclass of these functions are the so-called *multiplicative functions.* These functions satisfy $f(1) = 1$ and

$$f(mn) = f(n)f(m) \text{ for } (m, n) = 1.$$

The Möbius function and Euler totient function are important examples of multiplicative functions. If f is a multiplicative function, then so is $g(n)$ with $g(n) = \sum_{d|n} f(d)$. Indeed, more generally, if f and g are two multiplicative functions, so is $(f \star g)(n) = \sum_{d|n} f(d)g(n/d)$, which is the so called *Dirichlet convolution* of f and g. If $n = p_1^{e_1} \cdots p_s^{e_s}$ is the prime factorisation of n and f is multiplicative, then we have

$$\sum_{d|n} f(d) = \prod_{j=1}^{s} \left(1 + f(p) + \cdots + f(p^{e_j})\right).$$

A closely related formula is the following one; if $\sum_{d=1}^{\infty} f(d)$ is absolutely convergent and f is multiplicative, then

$$\sum_{d=1}^{\infty} f(d) = \prod_{p} \left(\sum_{j=0}^{\infty} f(p^j)\right). \tag{2}$$

The latter identity is known as *Euler's product identity.*

If f is multiplicative, then it is an easy observation that

$$f(m)f(n) = f((m, n))f([m, n]) \tag{3}$$

for all integers m and n, where (m, n) denotes the greatest common divisor of m and n and $[m, n]$ the lowest common multiple. Likewise we define $(m_1, m_2, \ldots, m_s)$ and $[m_1, m_2, \ldots, m_s]$, were we put $(m_1) = 1$ and $[m_1] = m_1$. Conversely, a function that satisfies the functional equation (3) is said to be *semi-multiplicative*. It can be shown [20] that if f is semi-multplicative, there exists a non-zero constant c, a positive integer a, and a multiplicative function f_1 such that

$$f(n) = \begin{cases} cf_1(n/a) & \text{if } a \nmid n; \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that if f is semi-multiplicative and k is a constant, then also $f(k/(k, n))$ is semi-multiplicative in n.

2.2 The Möbius function

Let n be an integer having prime divisors $p_1, \cdots, p_s$. Then the Möbius μ function is defined as follows:

$$\mu(n) = \begin{cases} 0 & \text{if } p^2|n \text{ for some prime } p; \\ (-1)^s & \text{otherwise.} \end{cases}$$

Note that $\mu(1) = 1$ and that $\mu(n)^2 = 1$ iff n is squarefree. The Möbius function satisfies the following identity:

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1; \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

An integer n is said to be *kth powerfree* if n is not divisible by a kth power of some integer > 1. Later we will need the following identity.

Proposition 1 *For any natural number k we have*

$$\sum_{d^k|n} \mu(d) = \begin{cases} 1 & \textit{if } n \textit{ is } k\textit{th power-free;} \\ 0 & \textit{otherwise.} \end{cases}$$

First proof. Using the multiplicativity of the Möbius-function one sees that the left hand side is a multiplicative function of n. It thus suffices to evaluate it when n is a prime power. Now

$$\sum_{d^k|p^\alpha} \mu(d) = \begin{cases} 1 & \text{if } \alpha \leq k-1; \\ 0 & \text{otherwise,} \end{cases}$$

from which the result follows. □

Second proof. Write $n = m^k r$, with the number r kth power free. Then the sum under consideration equals $\sum_{d|m} \mu(d)$. The result then follows by (4). □

An important formula involving the Möbius function is the celebrated *Möbius inversion formula*. It states that $f(n) = \sum_{d|n} g(d)$ iff $g(n) = \sum_{d|n} \mu(d) f(n/d)$. The Möbius function often arises in combinatorial problems, we will see several examples in this M.Sc. thesis. In Section 6 some auxiliary functions involving the Möbius function are considered.

2.3 The Euler totient function

The Möbius function often arises in combinatorial problems where so-called inclusion and exclusion is being used in counting. We demonstrate this by deriving an identity for Euler's totient function $\varphi(n)$, which is defined as

$$\varphi(n) = \sum_{1 \le j \le n,\ (j,n)=1} 1.$$

Proposition 2 *Let $N(d)$ denote the number of integers $1 \le m \le n$ that are divisible by d. Then*

$$\varphi(n) = \sum_{d|n} \mu(d) N(d) = n \sum_{d|n} \frac{\mu(d)}{d} = n \prod_{p|n} \left(1 - \frac{1}{p}\right).$$

Proof. Suppose $1 \le m \le n$ is an integer. In the expression $\sum_{d|n} \mu(d)N(d)$, the integer m is counted in those $N(d)$ for which both $d|m$ and $d|n$. It is counted with weight $\sum_{d|n,\ d|m} \mu(d)$. But, using (4), we see that

$$\sum_{d|n,\ d|m} \mu(d) = \sum_{d|(n,m)} \mu(d) = \begin{cases} 1 & \text{if } (n,m) = 1; \\ 0 & \text{otherwise} \end{cases}.$$

Thus $\phi(n) = \sum_{d|n} \mu(d)N(d)$. In case $d|n$, then $N(d) = n/d$. On using that μ is a multiplicative function, the result then follows. □

From Proposition 2 we easily infer that ϕ is a multiplicative function. Alternatively this can be easily deduced on using the Chinese remainder theorem.

For later use we note that by Proposition 2 we have

$$\frac{\varphi(\delta n)}{\varphi(n)} = \delta \frac{\prod_{p|n\delta}(1 - \frac{1}{p})}{\prod_{p|n}(1 - \frac{1}{p})} = \delta \prod_{\substack{p|\delta \\ p\nmid n}} (1 - \frac{1}{p}) = \delta \frac{\varphi(\delta_1)}{\delta_1}, \tag{5}$$

where $\delta_1 = \prod_{p|\delta,\ p\nmid n} p$. Note that as n runs over all integers, δ_1 will run over all squarefree divisors of δ. In particular, (5) implies that

$$\frac{\varphi(2n)}{\varphi(n)} = \begin{cases} 2 & \text{if } n \text{ is even;} \\ 1 & \text{otherwise.} \end{cases}$$

2.4 Roots of unity

An nth root of unity is a solution of $z^n = 1$ in $\mathbb{C}$. Note there are precisely n solutions; $e^{2\pi i/n}, \cdots, e^{2\pi in/n}$, with, of course, $e^{2\pi in/n} = 1$. Instead of $e^{2\pi ik/n}$ one often writes ζ_n^k with $\zeta_n = e^{2\pi i/n}$. Indeed, throughout this M.Sc. thesis we adopt the notation

$$\zeta_n = e^{\frac{2\pi i}{n}}.$$

An nth root of unity is said to be a *primitive* root of unity if it is of the form ζ_n^k with k and n coprime (which we will denote by $(k,n) = 1$). Such a primitive root

of unity has the property that it does not satisfy an identity of the form $z^m = 1$ with m a divisor $< n$ of n.

`Example`. For $n = 6$, ζ_6 and $\zeta_6^5 = \bar{\zeta}_6$ are the primitive roots of unity. Note that ζ_6^2 is not a primitive 6th order root of unity as it satisfies the identity $z^3 = 1$.

Note that there are $\varphi(n)$ primitive nth roots of unity. The roots of unity form a cyclic group of order n. The primitive roots of unity correspond with the generators of this group.

3 Ramanujan sums

The Ramanujan sum $c_n(m)$ is defined by

$$c_n(m) = \sum_{\substack{1\le k\le n\\(k,n)=1}} e^{\frac{2\pi imk}{n}} = \sum_{\substack{1\le k\le n\\(k,n)=1}} \zeta_n^{mk}.$$

Although Ramanujan was not the first to work with Ramanujan sums, he was the first tor realize their importance and use them consistently (especially in the theory of represenation of numbers as sum of squares, see e.g [8, Chapter IX]). The next proposition lists some of the important properties of Ramanujan sums. (These properties are all known or well-known except perhaps the formula in part 6, for which we do not have a reference.) By $\nu_p(n)$ we will denote the exponent of p in n, that is we have $\nu_p(n) = r$ iff $p^r|n$ and $p^{r+1} \nmid n$.

Proposition 3

1) *We have*

$$c_n(m) = \sum_{d|(n,m)} d\mu(\frac{n}{d}).$$

2) *We have*

$$c_n(m) = \mu\left(\frac{n}{(n,m)}\right)\frac{\varphi(n)}{\varphi(\frac{n}{(n,m)})}.$$

3) *We have* $-(n,m) \le c_n(m) \le (n,m)$ *and* $-\varphi(n) \le c_n(m) \le \varphi(n)$.

4) *We have* $c_n(m) \in \mathbb{Z}$.

5) *We have* $c_{n_1n_2}(m) = c_{n_1}(m)c_{n_2}(m)$ *if* $(n_1, n_2) = 1$*; i.e.* $c_n(m)$ *is multiplicative with respect to* n.

6) *The function* $c_n(m)$ *is semi-multiplicative in* m. *For fixed squarefree* n, *the function* $\mu(n)c_n(m)$ *is multiplicative in* m. *For arbitrary natural numbers* n *and* m *we have*

$$c_n(m) = \prod_{p|m} \mu\left(\frac{p^{\nu_p(n)}}{(n, p^{\nu_p(m)})}\right)\frac{\varphi(n)}{\varphi(\frac{n}{(n,p^{\nu_p(m)})})}.$$

7) *The following orthogonality relations hold (when* $r_1|r$ *and* $r_2|r$*):*

$$\frac{1}{r}\sum_{m=1}^{r} c_{r_1}(m)c_{r_2}(m) = \begin{cases} 0 & \text{if } r_1 \ne r_2; \\ \varphi(r_1) & \text{if } r_1 = r_2. \end{cases}$$

Proof. 1) Consider $g(n) := \sum_{1\le k\le n} e^{\frac{2\pi imk}{n}}$. Clearly this expression equals n if $n|k$ and zero otherwise. Expressing $g(n)$ in terms of Ramanujan sums we obtain $g(n) = \sum_{d|n} c_{n/d}(m)$, which by Möbius inversion gives $c_n(m) = \sum_{d|n} g(d)\mu(n/d)$. On using that $g(d) = d$ if $d|m$ and vanishes otherwise, the stated formula follows.
2) This result, due to O. Hölder, follows from property 1 on using (5) (see e.g. Exercise 1.1.14 of [16]).
3) This is a consequence of property 2 and formula (5).
4) First proof. Immediate by property 1. Second proof. The Ramanujan sum can be considered as an element of the field extension $\mathbb{Q}(\zeta_n)$. It is invariant under each of the automorphisms $\zeta_n \to \zeta_n^k$ with k coprime to n. Hence it is in the fixed field of $\mathbb{Q}(\zeta_n)$, which is $\mathbb{Q}$, since $\mathbb{Q}(\zeta_n) : \mathbb{Q}$ is Galois. Since ζ_n is an algebraic integer, the Ramanujan sum must even be an integer.
5) First proof. We have

$$c_{n_1}(m)c_{n_2}(m) = \sum_{\substack{1\le k_1\le n_1\\(k_1,n_1)=1}} e^{\frac{2\pi imk_1}{n_1}} \sum_{\substack{1\le k_2\le n_2\\(k_2,n_2)=1}} e^{\frac{2\pi imk_2}{n_2}} = \sum_{k_1,k_2} e^{\frac{2\pi im(k_1n_2+k_2n_1)}{n_1n_2}} = c_{n_1n_2}(m),$$

where we use the observation that the set of congruences classes of the form $k_1n_2 + k_2n_1$, with $1 \le k_j \le n_j$ and $(k_j, n_j) = 1$ for $j = 1$ and $j = 2$, consists of $\varphi(n_1n_2)$ distinct congruence classes mod n_1n_2, all of them coprime with n_1n_2.
Second proof. The assumptions on n_1 and n_2 imply that $(n_1n_2, m) = (n_1, m)(n_2, m)$. Now note that

$$c_{n_1n_2}(m) = \sum_{d|(n_1n_2,m)} d\mu(\frac{n_1n_2}{d}) = \sum_{\substack{d_1|(n_1,m)\\d_2|(n_2,m)}} d_1d_2\mu(\frac{n_1}{d_1}\frac{n_2}{d_2}) = c_{n_1}(m)c_{n_2}(m),$$

where in the last step we use the multiplicativity of μ.
6) The semi-multiplicativity is immediate from what has been said in Section 2.1 and property 2. The second assertion is an easy consequence of property 2 and the observation that if f is multiplicative, then $f(n)/f(n/(n,k))$ is a multiplicative function in k. The proof of the third assertio uses in addition to the previous proof ingredients the multiplicativity of μ.
7) For a proof see e.g. p. 17 of [22]. □

An arithmetic function f is said to be an *even* function of (n,r) if $f((n,r),r) = f(n,r)$ for all n (for a survey see [4]). By Property 2 $c_n(m) = c_n((n,m))$ and hence Ramanujan sums are even. By Property 4 we can alternatively define $c_n(m) = \sum_{1\le k\le n,\ (k,n)=1} \cos(2\pi mk/n)$.

An optimality property of Ramanujan sums was discovered by Bachman [2]. If $r \ge 1$ is any real number, he showed that for any sequence of real numbers b_k we have

$$\sum_{m=1}^{n} |\sum_{\substack{1\le k\le n\\(k,n)=1}} b_k e^{2\pi imk/n}|^r \ge \left(\frac{|\sum_{1\le k\le n,\ (k,n)=1} b_k|}{\varphi(n)}\right)^r \sum_{m=1}^{n} |c_n(m)|^r.$$

It follows that if we consider the infimum of the left handside over all sequences b_k with b_k this is assumed in case $b_k = 1$ for all k.

For some further properties of Ramanujan sums, such as for example the Brauer-Rademacher identity, we refer to Chapter 2 of [10].

3.1 Density and order of growth

If S is a set of natural numbers, by $\delta(S)$ we denote the limit of $x^{-1}\sum_{n\le x,\ n\in S} 1$ (as x tends to infinity) if this exists. Let $\pi(x)$ denote the number of primes $p \le x$. Recall that the prime number theorem asserts that asymptotically $\pi(x) \sim x/\log x$ (for a much stronger version see Lemma 3 below). If S is a set of prime numbers, by $\delta(S)$ we denote the limit of $\pi(x)^{-1}\sum_{p\le x,\ p\in S} 1$ (as x tends to infinity) if this exists.

If f and g are two functions defined on a set S, then we shall write $f(x) = O(g(x))$ (we say f is of *order* g) if the ratio $|f(x)/g(x)|$ is bounded for all $x \in S$. If the ratio $f(x)/g(x)$ tends to zero for x tending to a specified value x_0 (which may be infinite), then we shall write $f(x) = o(g(x))$. The symbols $O(\cdot)$ and $o(\cdot)$ are commonly called *Landau symbols*.

3.2 Ramanujan expansions of arithmetic functions

Ramaujan sums play a key role in the theory of *Ramanujan expansions*. For completeness we discuss this topic, though this will be not needed in the rest of this thesis.

Let

$$M(f) = \lim_{x\to\infty} \frac{1}{x}\sum_{n\le x} f(n).$$

denote the mean-value of $f : \mathbb{N} \to \mathbb{C}$ (if it exists). The Ramanujan sums have the following orthogonality property which is easily deduced from Property 7 of Proposition 3:

$$M(c_r \cdot c_s) = \begin{cases} \varphi(r) & \text{if } r = s; \\ 0 & \text{otherwise.} \end{cases}$$

On certain spaces of arithmetic functions there is an inner-product $\langle f \cdot g\rangle = M(f \cdot \bar{g})$ and this product suggests *Ramanujan expansions* $f \sim \sum_{r\ge1} a_r(f)c_r$ for arithmetic functions f in a suitable Hilbert space with coefficients $a_r(f) = \langle f \cdot c_r\rangle/\varphi(r)$. Indeed, Ramanujan showed that for example

$$\frac{\varphi(n)}{n} = \frac{6}{\pi^2}\sum_{r\ge1}\frac{\mu(r)c_r(n)}{\prod_{p|r}(p^2-1)},$$

$$\sigma_s(n) = \sum_{d|n} d^s = \frac{n^s}{\zeta(s)}\sum_{r=1}^{\infty}\frac{\mu(r)c_r(n)}{k^s\prod_{p|k}(1-1/p^s)} \quad (\mathrm{Re}(s) > 1),$$

and

$$r(n) = \pi\sum_{r\ge1}\frac{(-1)^{r-1}c_{2r-1}(n)}{2r-1},$$

where $r(n)$ denotes the number of representations of n as a sum of two squares and $\zeta(s)$ as usual the Riemann zeta function (for $\mathrm{Re}(s) > 1$ we have $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ and for all other $s \ne 1$ in the complex plane $\zeta(s)$ can be uniquely defined by analytic continuation). Rearick [20] proved that if F is an arithmetical function and an associated function f is defined by $F(n) = \sum_{d|n} df(d)$ such that $\sum_k\sum_n |f(kn)| < \infty$, then we have, with $a(n) = \sum_k f(kn)$, that the series

$\sum_k a(k)c_k(n)$ converges absolutely for each n and we have $F(n) = \sum_k a(k)c_k(n)$. For a readable survey on Ramanujan expansions, see e.g. [21], for more detailed information see [22].

3.3 Ramanujan sums and Dirichlet series

Given an arithmetic function f an important quantity in analytic number theory is the associated generating series, $\sum_{n=1}^{\infty} f(n)/n^s$, which is usually named after Dirichlet. If we take $f(n) = c_n(m)$, we obtain

$$R_m(s) = \sum_{n=1}^{\infty} \frac{c_n(m)}{n^s}.$$

By property 1 of Proposition 3 we have $c_n(m) = \sum_{dr=n,\ d|m} \mu(r)d$ and hence

$$\frac{c_n(m)}{n^s} = \sum_{dr=n,\ d|m} \frac{\mu(r)}{r^s} d^{1-s}$$

and so

$$R_m(s) = \sum_{n=1}^{\infty} \sum_{dr=n,\ d|m} \frac{\mu(r)}{r^s} d^{1-s} = \sum_{d|m} d^{1-s} \sum_{r=1}^{\infty} \frac{\mu(r)}{r^s} = \frac{\sigma_{1-s}(m)}{\zeta(s)} = \frac{m^{1-s}\sigma_{s-1}(m)}{\zeta(s)}.$$

(This proof is due to Estermann, for Ramanujan's (longer) proof see [8, p. 140]). Taking $s = 2$ we obtain in particular

$$\sigma_1(m) = \sum_{d|m} d = \frac{\pi^2}{6} m \left(1 + \frac{(-1)^m}{2^2} + \frac{2\cos(2m\pi/3)}{3^2} + \frac{2\cos(m\pi/4)}{4^2} + \cdots \right),$$

which shows in a striking way the oscillation of $\sigma_1(m)$ about its 'average' $\pi^2 m/6$. (It is not difficult to show that $\sum_{m\le x} \sigma_1(m) \sim \frac{\pi^2}{6} \sum_{m\le x} m \sim \frac{\pi^2}{12} x^2$ as x tends to infinity:

$$\sum_{m\le x} \sigma_1(m) = \sum_{rd\le x} d = \frac{1}{2} \sum_{r\le x} [\frac{x}{r}]([\frac{x}{r}] + 1) = \frac{1}{2} \sum_{r\le x} \frac{x^2}{r^2} + O\left(x \sum_{r\le x} \frac{1}{r}\right),$$

for which it follows that $\sum_{m\le x} \sigma_1(m) = \pi^2 x^2/12 + O(x \log x)$.)

Similarly one can consider $R_n(s) = \sum_{m=1}^{\infty} c_n(m)/m^s$. Invoking property 1 of Proposition 3 again, one easily obtains that $R_n(s) = \zeta(s) \sum_{d|n} \mu(n/d) d^{1-s}$.

In §7 the Dirichlet series $f_m^{(j)}(s) = \sum_{n=1}^{\infty} c_n(m)^j/n^s$ features (with j a non-negative integer). Define

$$F_m^{(j)}(s) = \prod_{p|m} \frac{\left(1 + \sum_{k=1}^{\nu_p(m)} \frac{(p^k - p^{k-1})^j}{p^{ks}} + \frac{(-p^{\nu_p(m)})^j}{p^{(\nu_p(m)+1)s}}\right)}{1 + \frac{(-1)^j}{p^s}}. \tag{6}$$

Using Properties 5 and 2 of Proposition 3 and Euler's product identity, one infers that, for $\mathrm{Re}(s) > 1$,

$$f_m^{(j)}(s) = F_m^{(j)}(s) \prod_p (1 + \frac{(-1)^j}{p^s}).$$

It follows that

$$f_m^{2j}(s) = F_m^{(2j)}(s)\zeta(s)/\zeta(2s) \text{ and } f_m^{2j-1}(s) = F_m^{(2j-1)}(s)/\zeta(s). \tag{7}$$

4 Cyclotomic polynomials

The nth cyclotomic polynomial, $\Phi_n(x)$, is defined by

$$\Phi_n(X) = \prod_{\substack{1\le j\le n\\(j,n)=1}} (X - \zeta_n^j). \tag{8}$$

Note that the roots of $\Phi_n(X)$ are precisely the primitive nth roots of unity. Clearly $\Phi_n(X)$ is a monic polynomial of degree $\varphi(n)$. We write

$$\Phi_n(X) = \sum_{k=0}^{\phi(n)} a_n(k)x^k.$$

It is the coefficients $a_n(k)$ that later will have our special attention. Using induction with respect to n it is not difficult to show that $a_n(k) \in \mathbb{Z}$ (see e.g. [16, Exercise 1.5.24]). Note that

$$X^n - 1 = \prod_{d|n} \prod_{\substack{1\le j\le n\\(j,n)=d}} (X - \zeta_n^j) = \prod_{d|n} \Phi_{\frac{n}{d}}(X) = \prod_{d|n} \Phi_d(X).$$

By applying Möbius inversion one infers from this that

$$\Phi_n(X) = \prod_{d|n} (X^d - 1)^{\mu(\frac{n}{d})}. \tag{9}$$

Thus, for example,

$$\Phi_p(X) = \frac{X^p - 1}{X - 1} = X^{p-1} + X^{p-2} + \cdots + X + 1.$$

Furthermore, it is known that $\Phi_{2n}(X) = \Phi_n(-X)$ if $n > 1$ is odd and $\Phi_n(X) = \Phi_{\gamma(n)}(X^{n/\gamma(n)})$, where $\gamma(n) = \prod_{p|n} p$ is the *squarefree kernel* of n. The latter two properties are the reason that some authors in this area restrict themselves to n squarefree and odd. A further important property is that for $n > 1$ we have $X^{\varphi(n)}\Phi_n(1/X) = \Phi_n(X)$. It is not difficult to infer these three properties from (9). In terms of the coefficients these properties imply (respectively):

$$a_n(k) = \begin{cases} a_{\gamma(n)}(\frac{k\gamma(n)}{n}) & \text{if } \frac{n}{\gamma(n)}|k; \\ 0 & \text{otherwise,} \end{cases} \tag{10}$$

$$a_{2n}(k) = (-1)^k a_n(k) \quad \text{for } n > 1,\ 2 \nmid n; \tag{11}$$

$$a_n(k) = a_n(\varphi(n) - k) \quad \text{for } n > 1.$$

It can be shown that $\Phi_n(X)$ is irreducible over $\mathbb{Q}$. This has as consequence that the degree of the cyclotomic field $\mathbb{Q}(\zeta_n)$ over $\mathbb{Q}$ equals $\varphi(n)$. Over finite fields

the cyclotomic fields factor in general. Indeed, over $\mathbb{F}_q$, $\Phi_d(X)$ has $\phi(d)/\mathrm{ord}_q(d)$ distinct irreducible factors if $(q,n)=1$. If $p\nmid m$, then $p|\Phi_m(a)$ if and only if the order of $a(\mathrm{mod}\ p)$ is m (Proposition 18 below). This can be used to infer the infinitude of primes $p\equiv 1(\mathrm{mod}\ m)$ [16, Exercise 1.5.30].

Note that the coefficient, $a_n(\varphi(n)-1)$, of $X^{\varphi(n)-1}$ equals

$$-\sum_{\substack{1\le j\le n\\(j,n)=1}}\zeta_n^j=-c_n(1)=-\mu(n),$$

where the last equality follows from Property 5 of Proposition 3. Since $a_n(\varphi(n)-1)=a_n(1)$ for $n>1$, we conclude that $a_n(1)=-\mu(n)$. Another connection between cyclotomic polynomials and Ramanujan sums is given by the following result due to Nicol [18].

Proposition 4 *We have*

$$\Phi_n(X)=\exp\left(-\sum_{m=1}^{\infty}\frac{c_n(m)}{m}X^m\right)$$

and

$$\sum_{m=1}^{n}c_n(m)X^{m-1}=(X^n-1)\frac{\Phi_n'(X)}{\Phi_n(X)}.$$

Using cyclotomic polynomials we can give an easy proof of Property 1 of Proposition 3. We take the logarithmic of the right hand side of (8) and compare its Taylor series with that obtained on taking the logarithm of the right hand side of (9). The details are left to the reader.

4.1 On the size of the coefficients of cyclotomic polynomials

The size of the coefficients of cyclotomic polynomials is a much researched topic. This presumably stems from wonder over the fact that so often the coefficients are in $\{0,\pm1\}$. Indeed, only for $n\ge 105$ some coefficients outside this range appear. For example, we have $a_{105}(7)=-2$. The amazement over the smallness of $a_n(m)$ was eloquently worded by D. Lehmer [9] who wrote: 'The smallness of $a_n(m)$ would appear to be one of the fundamental conspiracies of the primitive nth roots of unity. When one considers that $a_n(m)$ is a sum of $\binom{\phi(n)}{m}$ unit vectors (for example 73629072 in the case of $n=105$ and $m=7$) one realizes the extent of the cancellation that takes place'.

Migotti showed in 1883 that $a_{pq}(i)\in\{0,\pm1\}$, with p and q odd primes. On the other hand Emma Lehmer showed that $a_{pqr}(i)$ can be arbitrarily large (with p,q and r odd primes). Sister Marion Beiter put forward in 1971 the conjecture that in fact $a_{pqr}(i)\le(p+1)/2$ for all i and for any $p<q<r$ and that this bound is best possible. This conjecture remains unsolved, for some recent progress see [3].

Let us put $B(k)=\max_{n\ge1}|a_n(k)|$. For small values of k, Möller [13] gave a method to compute $B(k)$. The next table (taken from [13]) gives some values of $B(k)$.

Table 2: Value of $B(k)$ for $1 \le k \le 30$

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$B(k)$	1	1	1	1	1	1	2	1	1	1	2	1	2	2	2
k	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$B(k)$	2	3	3	3	3	3	3	4	3	3	3	3	4	4	5

The table suggests that $B(k) \ge k$ for every $k \ge 1$, this is however very far from the case: Bachman [1] extended work of several earlier authors and showed that

$$\log B(k) = C_0 \frac{\sqrt{k}}{(\log k)^{1/4}} \left(1 + O\left(\frac{\log\log k}{\sqrt{\log k}}\right)\right). \tag{12}$$

In the opposite direction, let us put $A(n) = \max_m |a_n(m)|$. Erdös has shown that there exists a $c > 0$ and that there are infinitely many n such that

$$\log A(n) \gg \exp\left(\frac{c \log n}{\log\log n}\right).$$

On the other hand it is known that

$$\log A(n) < \exp\left((\log 2 + o(1))\frac{\log n}{\log\log n}\right),$$

where the constant $\log 2$ is best possible. So the conspiration of the primitive roots of unity is not always that efficient !

Jiro Suzuki [23] proved the following beautiful result, which we present with a slightly shortened proof. Proposition 17, using a slightly more involved argument, strengthens Suzuki's result.

Proposition 5 *We have* $\{a_n(k) : n, k \in \mathbb{N}\} = \mathbb{Z}$.

Proof. Given any integer $s \ge 2$ it is a consequence of the prime number theorem (cf. the proof of Proposition 9) that it is possible to find primes $2 < p_1 < p_2 < \cdots < p_s$ such that $p_1 + p_2 > p_s$. Let q be any prime $> p_s$. If s is even, let $m = p_1 \cdots p_s q$, otherwise let $m = p_1 \cdots p_s$. We claim that

$$\Phi_m(X) = \sum_{j=0}^{p_s - 1} a_m(j) X^j + (1 - s) X^{p_s} \pmod{X^{p_s+1}}. \tag{13}$$

The claim shows that $a_m(p_s) = 1 - s$. By (11) we have $a_{2m}(p_s) = s - 1$. On noting that $a_4(1) = 0$ the result follows. We next prove (13). By (24) and the observation that $\{d : d|m,\ d < p_s + 1\} = \{1, p_1, \ldots, p_s\}$ and that for $i \ne j$, $p_i + p_j \ge p_1 + p_2 > p_s$, we infer that mod X^{p_s+1} we have

$$\begin{aligned}
\Phi_m(X) &\equiv (1 - X^{p_1}) \cdots (1 - X^{p_s})/(1 - X) \pmod{X^{p_s+1}} \\
&\equiv (1 + X + \cdots + X^{p_s - 1})(1 - X^{p_1} - \cdots - X^{p_{s-1}}) \pmod{X^{p_s+1}} \\
&\equiv \sum_{j=0}^{p_s-1} a_m(j) X^j + (1 - s) X^{p_s} \pmod{X^{p_s+1}}.
\end{aligned}$$

This concludes the proof. $\square$

The basic idea of this proof seems to have been first formulated by I. Schur in a letter to E. Landau. He used it to show that $a_n(k)$ can be arbitrarily large.

A very readable survey on properties of coefficients of cyclotomic polynomials is provided by [24].

In this thesis we are especially interested in how often certain values are assumed by $a_n(k)$ for small fixed k (§8), where n either runs over all integers n or over the numbers of the form $p-1$ with p a prime (§9.3).

5 Primitive roots

We already stated that for every prime p there exists a primitive root g mod p and that all other primitive roots mod p are of the form g^j with j coprime to $p-1$. Since $g, g^2, ..., g^{p-1}$ are all distinct mod p, there are precisely $\varphi(p-1)$ distinct primitive roots mod p.

Emil Artin conjectured in 1927 that for every integer g not equal to -1 or a square, there are infinitely many primes p such that g is a primitive root. More precisely, he conjectured that the set of primes p for which g is a primitive root has a density which should equal a rational multiple of A, the Artin constant, which is defined as

$$A = \prod_p (1 - \frac{1}{p(p-1)}) = 0.3739558136\cdots$$

Artin's conjecture was proved in 1967 under the assumption of the Generalized Riemann Hypothesis (GRH) by C. Hooley. In our consideration of the problem stated in the introduction, the constant A will also arise. Artin's constant (and indeed many similar ones) can be evaluated with many decimals of precision by expanding it in terms of values of the Riemann zeta-function evaluated at integers exceeding one, see e.g. [15]. We return to the Artin primitive root conjecture in Section 6.2.2.

6 Some sums involving the Möbius function

Two important auxiliary quantities that will arise are, for given positive integers k, r and j, are the sums

$$\sum_{\substack{n \le x \\ (n,r)=1}} \mu(n)^j \text{ and } \sum_{\substack{p \le x \\ (p-1,k)=r}} \mu(p-1)^j. \tag{14}$$

A number of the form $p-t$ with t fixed is said to be a *shifted prime*. Note that we can restrict to the case where $j=1$ and $j=2$. The case $j=2$ can be dealt with by fairly elementary analytic number theory. The case $j=1$ is rather harder, in the case of shifted primes providing a non-trivial estimate would be considered a major achievement by the experts we consulted.

6.1 Counting squarefree integers

In this section we are concerned with estimating the first sum appearing in (14).

Lemma 1 *Let $r \geq 1$ be an integer. We have*

$$\sum_{\substack{m\leq x\\(m,r)=1}} \mu(m)^2 = \frac{6x}{\pi^2\prod_{p|r}(1+\frac{1}{p})} + O(\sqrt{x}\varphi(r)),$$

where the implied constant is absolute.

Proof. We have, by inclusion and exclusion,

$$\sum_{\substack{m\leq x\\(m,r)=1}} \mu(m)^2 = \sum_{\substack{d\leq\sqrt{x}\\(d,r)=1}} \mu(d)A_r(\frac{x}{d^2}),$$

where $A_r(x)$ denotes the number of integers $n \leq x$ that are coprime with r. Note that

$$[\frac{x}{r}]\varphi(r) \leq A_r(x) \leq [\frac{x}{r}]\varphi(r) + \varphi(r)$$

and hence $A_r(x) = \varphi(r)x/r + O(\varphi(r))$. On using the latter estimate we obtain

$$\begin{aligned}\sum_{\substack{m\leq x\\(m,r)=1}} \mu(m)^2 &= x\frac{\varphi(r)}{r}\sum_{\substack{d\leq\sqrt{x}\\(d,r)=1}}\frac{\mu(d)}{d^2} + O(\sqrt{x}\varphi(r)).\\ &= x\frac{\varphi(r)}{r}\sum_{(d,r)=1}^{\infty}\frac{\mu(d)}{d^2} + O(\sqrt{x}\varphi(r)).\\ &= \frac{6x}{\pi^2\prod_{p|r}(1+\frac{1}{p})} + O(\sqrt{x}\varphi(r)),\end{aligned}$$

where we used that

$$\frac{\varphi(r)}{r}\sum_{(d,r)=1}^{\infty}\frac{\mu(d)}{d^2} = \frac{\varphi(r)}{r}\prod_{p\nmid r}(1-\frac{1}{p^2}) = \frac{\varphi(r)}{\zeta(2)r\prod_{p|r}(1-\frac{1}{p^2})} = \frac{1}{\zeta(2)\prod_{p|r}(1+\frac{1}{p})}$$

and $\zeta(2) = \pi^2/6$. □

We also need to estimate $M_r(x) = \sum_{\substack{m\leq x\\(m,r)=1}} \mu(m)$. Estimating this quantity turns out to be much harder. The quantity $M_1(x)$ is the summatory function of the Möbius function and this function is called the *Mertens function*. The prime number theorem is known to be equivalent with the assertion that $\lim_{x\to\infty} M_1(x)/x = 0$. The celebrated Mertens' conjecture states that $|M_1(x)| < \sqrt{x}$ for every x. If this conjecture would hold, then from the easily proved equality

$$\frac{1}{\zeta(s)} = s\int_1^{\infty}\frac{M_1(x)}{x^{s+1}}ds,$$

it would follow that $\zeta(s)$ has no zeros with $\mathrm{Re}(s) > 1/2$, i.e. the Riemann Hypothesis would follow. In fact, the Riemann Hypothesis is known to be equivalent

with the assertion that $M_1(x) = O(x^{1/2+\epsilon})$, for every $\epsilon > 0$. In 1986 Odlyzko and te Riele disproved the Mertens conjecture. The behaviour of $M_1(x)$ is thus closely related to the zero free region of the Riemann zeta function. The largest known zero free region of $\zeta(s)$, due to Korobov, was used by Walfisz to show that

$$M_1(x) = O(x \exp(-C(\log x)^{3/5}(\log\log x)^{-1/5})).$$

In this work we will be satisfied with merely determining the average of $M_r(x)$:

Lemma 2 *We have*

$$\sum_{\substack{m \le x \\ (m,r)=1}} \mu(m) = o(x),$$

where the implied constant may depend on r.

To prove the lemma, we will apply the Wiener-Ikehara Tauberian theorem in the following form.

Theorem 1 *Let $f(s) = \sum_{n=1}^{\infty} a_n/n^s$ be a Dirichlet series. Suppose there exists a Dirichlet series $F(s) = \sum_{n=1}^{\infty} b_n/n^s$ with positive real coefficients such that*
(a) $|a_n| \le b_n$ *for all n;*
(b) *the series $F(s)$ converges for $Re(s) > 1$;*
(c) *the function $F(s)$ can be extended to a meromorphic function in the region $Re(s) \ge 1$ having no poles except for a simple pole at $s = 1$.*
(d) *the function $f(s)$ can be extended to a meromorphic function in the region $Re(s) \ge 1$ having no poles except possibly for a simple pole at $s = 1$ with residue r.*
Then

$$\sum_{n \le x} a_n = rx + o(x), \;\; x \to \infty.$$

In particular, if $f(s)$ is holomorphic at $s = 1$, then $r = 0$ and $\sum_{n \le x} a_n = o(x)$ as $x \to \infty$.

Proof of Lemma 2. We apply the Wiener-Ikehara theorem with $F(s) = \zeta(s)$ and

$$f(s) = \sum_{(n,r)=1} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)\prod_{p|r}(1 - \frac{1}{p^s})}.$$

Of course $F(s)$ satisfies the required properties and has a simple pole at $s = 1$ with residue one. Since the finite product in the formula for $f(s)$ is regular for $\mathrm{Re}(s) > 0$, the result follows on using the well-known fact that $1/\zeta(s)$ can be extended to a meromorphic function in the region $\mathrm{Re}(s) \ge 1$ (and hence $r = 0$). □

6.2 Counting squarefree shifted primes

In this section we are concerned with estimating the second sum in (14).

6.2.1 On a result of Mirsky

Theorem 2 below is due to Mirsky [12] (with the difference that in his version the O-constant depends at most on k, r and H). In his paper Mirsky states two theorems, of which he only proves the first (the proof of the second being similar). Mirsky's second theorem is stated below. For completeness we give the proof here. Recall that the letters p and q are used to indicate primes.

Theorem 2 *Let r be any non-zero integer, k any integer greater than 1, and H any positive number. Then*

$$\#\{q \le x : q - r \text{ is } k\text{−free}\} = \prod_{p \nmid r}\left(1 - \frac{1}{p^{k-1}(p-1)}\right)\mathrm{Li}(x) + O\left(\frac{x}{\log^H x}\right),$$

where the O-constant depends at most on k and H.

Our proof rests on the Siegel-Walfisz theorem:

Lemma 3 [19, Satz 4.8.3]. *Let $\pi(x; m, l)$ denote the number of primes $q \le x$ with $q \equiv l(\mathrm{mod}\ m)$. Let $C > 0$ be arbitrary. Then*

$$\pi(x; m, l) = \frac{\mathrm{Li}(x)}{\varphi(m)} + O(xe^{-c_1\sqrt{\log x}}),$$

uniformly for $1 \le m \le \log^C x$, $(l, m) = 1$, where the constants depend at most on C.

Note that if $(l, m) > 1$ there is at most one prime $p \equiv l(\mathrm{mod}\ m)$. If $(l.m) = 1$, then the above result implies that $\delta(p \equiv l(\mathrm{mod}\ m)) = 1/\varphi(m)$, thus asymptotically the primes are equidistributed over the primitive congruence classes mod m.

Proof of Theorem 2. The dependence of O-constants on r, k and H is indicated by using them as index, thus O_H means that the implied constant depends at most on H.

Let $y = \log^H x$. By Proposition 1 we have

$$\begin{aligned}\#\{q \le x : q - r \text{ is } k\text{−free}\} &= \sum_{q \le x}\sum_{a^k | q-r} \mu(a) \\ &= \sum_{a \le y}\mu(a) \sum_{\substack{q \le x \\ q \equiv r(\mathrm{mod}\ a^k)}} 1 + \sum_{a > y}\mu(a) \sum_{\substack{q \le x \\ q \equiv r(\mathrm{mod}\ a^k)}} 1 \\ &= I_1 + I_2, \end{aligned} \tag{15}$$

say. We have, using Lemma 3 with $C = kH$,

$$\begin{aligned} I_1 &= \sum_{\substack{a \le y \\ (a,r)=1}} \mu(a) \sum_{\substack{q \le x \\ q \equiv r(\mathrm{mod}\ a^k)}} 1 + O(y) \\ &= \sum_{\substack{a \le y \\ (a,r)=1}} \mu(a)\Big\{\frac{\mathrm{Li}(x)}{\varphi(a^k)} + O_H(\frac{x}{\log^{2H} x})\Big\} + O(y) \end{aligned}$$

$$= \sum_{\substack{a\le y\\(a,r)=1}} \frac{\mu(a)}{\varphi(a^k)}\mathrm{Li}(x) + O_H(\frac{xy}{\log^{2H} x}) + O(y). \tag{16}$$

Clearly

$$\sum_{\substack{a\le y\\(a,r)=1}} \frac{\mu(a)}{\varphi(a^k)} = \sum_{\substack{a=1\\(a,r)=1}}^{\infty} \frac{\mu(a)}{\varphi(a^k)} + O\left(\sum_{a\ge y}\frac{1}{\varphi(a^k)}\right). \tag{17}$$

By Euler's product identity we have

$$\sum_{\substack{a=1\\(a,r)=1}}^{\infty} \frac{\mu(a)}{\varphi(a^k)} = \prod_{p\nmid r}\left(1-\frac{1}{p^{k-1}(p-1)}\right). \tag{18}$$

Using that $k \ge 2$ and the classical estimate $\varphi(m) \gg m/\log\log m$, we obtain

$$\sum_{a\ge y}\frac{1}{\varphi(a^k)} = O\left(\sum_{a\ge y}\frac{\log(k\log a)}{a^k}\right) = O_k\left(\frac{\log\log y}{y^{k-1}}\right) = O_k\left(\frac{\log\log y}{y}\right). \tag{19}$$

On combining (16) with (17), (18) and (19), we infer that

$$I_1 = \prod_{p\nmid r}\left(1-\frac{1}{p^{k-1}(p-1)}\right)\mathrm{Li}(x) + O_k\left(\frac{x\log\log y}{(\log x)y}\right) + O_H(\frac{xy}{\log^{2H} x}) + O(y).$$

Note that

$$|I_2| \le \sum_{a>y}\ \sum_{\substack{m\le x\\ m\equiv r(\mathrm{mod}\ a^k)}} 1 \le \sum_{a>y}[\frac{x}{a^k}] = O(\frac{x}{y^{k-1}}) = O(\frac{x}{y}),$$

where in the second sum the summation is over all integers $m \le x$ satisfying $m \equiv r(\mathrm{mod}\ a^k)$. On adding the estimates for I_1 and $|I_2|$, the result follows from (15). □

6.2.2 Connection with Artin's primitive root conjecture

By Theorem 2 the density of primes p such that $p-1$ is squarefree equals A, the Artin constant. The Artin constant also arose in Section 5 in the context of primitive roots. This raises the question whether there is some connection between the two problems. We will now show that this is indeed the case; both problems are in fact special cases of a generalization of Artin's primitive root conjecture. We assume some familiarity with algebraic number theory.

We recall that a prime p splits completely in $\mathbb{Q}(\zeta_n)$ iff $p \equiv 1(\mathrm{mod}\ n)$. Note that $p-1$ is squarefree iff $p \not\equiv 1(\mathrm{mod}\ q^2)$ with q any prime. It follows that $p-1$ is squarefree iff p does not split completely in any of the fields $\mathbb{Q}(\zeta_{q^2})$ with q a prime. It is a consequence of Chebotarev's density theorem that the density of primes q that split completely in a normal extension $K:\mathbb{Q}$ equals $1/[K:\mathbb{Q}]$, where $[K:\mathbb{Q}]$ denotes the degree of the extension. On noting that the compositum of $\mathbb{Q}(\zeta_{q_1^2}), ..., \mathbb{Q}(\zeta_{q_s^2})$, where $q_1, ..., q_s$ are distinct primes, equals $\mathbb{Q}(\zeta_{(q_1q_2\cdots q_s)^2})$, we

expect by inclusion and exclusion that the density of primes p for which $p-1$ is squarefree equals

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{[\mathbb{Q}(\zeta_{n^2}):\mathbb{Q}]}\ (=\sum_{n=1}^{\infty}\frac{\mu(n)}{\varphi(n^2)}=A).$$

Indeed, the primes p for which a given integer g with $g\neq -1$ or a square is a primitive root mod p can be described in a similar way. Here we want that for each prime q with $q|p-1$ we have that $g^{\frac{p-1}{q}}\not\equiv 1(\text{mod }p)$, which can be reformulated as p does not split completely in $\mathbb{Q}(\zeta_q,g^{1/q})$ for any prime q. By inclusion and exclusion we expect then that that the density of primes p for which g is a primitive root equals

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{[\mathbb{Q}(\zeta_n,g^{1/n}):\mathbb{Q}]}. \tag{20}$$

In fact Hooley proved in 1967 that the number given in (20) is indeed the correct density, under the assumption of GRH. For squarefree n the degree $[\mathbb{Q}(\zeta_n,g^{1/n}):\mathbb{Q}]$ is generically equal to $n\varphi(n)$, which is the exact degree of $[\mathbb{Q}(\zeta_{n^2}):\mathbb{Q}]$ and hence the arisal of A in both problems does not surprise us.

Both problems considered here are in fact special cases of the following generalization of Artin's primitive root problem. This generalization was first studied by Goldstein and later by M. Ram Murty [17]. Let K be an algebraic number field. Let $\mathcal{F}$ be a family of number fields, normal and of finite degree over K. Determine the number of prime ideals $\mathcal{P}$ of K such that $N_{K/\mathbb{Q}}(\mathcal{P})\leq x$ and which do not split completely in any element $\neq K$ of $\mathcal{F}$. On taking $K=\mathbb{Q}$ and F to be the set of fields of the form $\mathbb{Q}(\zeta_q,g^{1/q})$, where q runs over the primes, we obtain Artin's primitive root problem. On taking $K=\mathbb{Q}$ and F to be the set of fields of the form $\mathbb{Q}(\zeta_{q^2})$, where q runs over the primes, we obtain the special case of Mirsky's result considered above.

From an heuristical point of view the arisal of Artin's constant in the squarefree problem is not surprising. The density of primes q such that $q\not\equiv 1(\text{mod }p^2)$ is $1-1/(p(p-1))$. Imposing the condition $q\not\equiv 1(\text{mod }p^2)$ for each prime p and assuming all the conditions to represent independent events, one arrives at A as the expected density of the primes q with $q-1$ squarefree. The independence of the various local conditions is a consequence of the fact that for any two distinct primes r and s we have $\mathbb{Q}(\zeta_{r^2})\cap\mathbb{Q}(\zeta_{s^2})=\mathbb{Q}$.

For the Artin problem we impose the local condition that q does not satisfy both $q\equiv 1(\text{mod }p)$ and $g^{(q-1)/p}\equiv 1(\text{mod }p)$. The density of primes q not satisfying the latter two conditions is $1-1/[\mathbb{Q}(\zeta_p,g^{1/p}):\mathbb{Q}]$. If the conditions for the various primes p would be independent, we would expect (as Artin original did) the density to equal

$$\prod_p\left(1-\frac{1}{[\mathbb{Q}(\zeta_p,g^{1/p}):\mathbb{Q}]}\right), \tag{21}$$

which if $g\neq -1,0,1$ equals a rational number times the Artin constant. However, it is not always true (with r and s as before) that $\mathbb{Q}(\zeta_r,g^{1/r})\cap\mathbb{Q}(\zeta_s,g^{1/s})=\mathbb{Q}$ (for example when $r=5$, $s=2$ and $g=5$ in which case the intersection equals $\mathbb{Q}(\sqrt{5})$) and hence we expect the density to come out different from (21) sometimes (as can be shown true assuming GRH).

6.2.3 A variation of Mirsky's result

We will need a variation of Mirsky's result for $k = 2$ and $r = 1$. Let $S = (q_1, \ldots, q_t)$ be a (possibly empty) finite sequence of primes satisfying $q_1 < q_2 < \ldots < q_t$. We write $\nu_S(n) = (\nu_{q_1}(n), \ldots, \nu_{q_t}(n))$. We say that a number n is S-squarefree if $q^2 \nmid n$ for every prime q with $q \notin S$. We put

$$\mu_S(n) = \begin{cases} (-1)^{\sum_{q|n,\ q\notin S} 1} & \text{if } n \text{ is } S\text{-squarefree;} \\ 0 & \text{otherwise.} \end{cases}$$

If S is the empty set, then $\mu_S(n) = \mu(n)$.

Proposition 6 *Let t be any non-zero integer, Let $S = \{q_1, \ldots, q_t\}$ and $e_1, \ldots, e_t$ be natural numbers. Write $Q = \prod_{j=1}^t q_j^{e_j}$. We have*

$$\sum_{\substack{p\le x \\ \nu_S(p-1)=(e_1,\ldots,e_t)}} \mu_S(p-1)^2 =$$

$$\frac{1}{Q}\prod_{\substack{1\le j\le t \\ e_j=0}}\left(1-\frac{1}{q_j-1}\right)\prod_{q\notin S}\left(1-\frac{1}{q(q-1)}\right)\mathrm{Li}(x) + O\left(\frac{x}{\log^H x}\right),$$

where the O-constant depends at most on $S, e_1, \ldots, e_t$ and H.

Proof. A variation of Theorem 2 and left to the reader. □

`Remark`. Again the constant appearing in this result is not surprising. For each prime q we impose a condition at our prime p occurring in the sum: if $q \in S$ we prescribe $\nu_q(p-1)$ and if $q \notin S$ we require that $p \not\equiv 1 (\mathrm{mod}\ q^2)$. If for fixed q we compute the densities of primes satisfying this condition and if we multiply all these 'local densities' together, we arrive at the density

$$\delta(\mu_S(p-1)\neq 0)\prod_{j=1}^t \delta(\nu_{q_j}(p-1)=e_j) =$$

$$\frac{1}{\varphi(Q)}\prod_{\substack{1\le j\le t \\ e_j=0}}\left(1-\frac{1}{q_j-1}\right)\prod_{\substack{1\le j\le t \\ e_j\ge 1}}\left(1-\frac{1}{q_j}\right)\prod_{q\notin S}\left(1-\frac{1}{q(q-1)}\right),$$

which equals the density given in Proposition 6

6.2.4 A conjecture

It is generally believed that primes p are such that $p-1$ does not have a preference with regards to having an even or odd number of prime factors. In other words, the following conjecture (thought to be deep by the experts) is generally believed: we have $\sum_{p\le x}\mu(p-1) = o(\pi(x))$, as x tends to infinity. We propose a slightly more general conjecture.

Conjecture 1 *Let t be any non-zero integer, Let $S = \{q_1, \ldots, q_t\}$ and $e_1, \ldots, e_t$ be natural numbers. We have*

$$\sum_{\substack{p \le x \\ \nu_S(p-1)=(e_1,\ldots,e_t)}} \mu_S(p-1) = o(\pi(x)),$$

where the o-constant depends at most on $S, e_1, \ldots, e_t$ and H.

All the numerical data we came across in this respect seemed to be not inconsistent with this conjecture.

7 Value distribution of Ramanujan sums

We discuss the value distribution of $c_n(m)$ as n varies over the integers and m is fixed. We do this following Aurel Wintner's paper [25]. Wintner argues that the prime number theorem is equivalent with the Möbius function having an asymptotic distribution function and notes that $c_n(1) = \mu(n)$. A natural question arising then is whether $c_n(m)$ for m fixed has an asymptotic distribution function and if so, what it looks like. We next recall some facts regarding moments and distribution functions.

7.1 Distribution functions

A sequence $\{\alpha_j\}_{j=0}^{\infty}$ is said to have an *asymptotic distribution function*, ρ, if there exists a monotone function $\rho(\alpha)$ such that

$$\lim_{x\to\infty} \frac{\sum_{\alpha_j \le x} 1}{x} = \rho(\alpha),$$

holds at every continuity point α of ρ and, moreover, we have $\rho(-\infty) = 0$ and $\rho(\infty) = 1$. It is known that if $\lim_{j\to\infty} \sup |\alpha_j| < \infty$, it has an asymptotic distribution ρ iff $M_n(\alpha_n^j)$ exists for every j; in which case

$$M_n(\alpha_n^j) = [\rho]_j = \int_{-\infty}^{\infty} \alpha^j d\rho(\alpha).$$

(We define $\alpha_n^0 = 1$ and hence $[\rho]_0 = 1$.) Given a distribution function $\rho(\alpha)$ we can consider, $L(u;\rho)$, the Fourier-Stieltjes transform of ρ:

$$L(u;\rho) = \int_{-\infty}^{\infty} e^{i\alpha u} d\rho(\alpha).$$

If the moments exist for every j, then

$$L(u;\rho) = \sum_{j=0}^{\infty} \frac{i^j [\rho]_j}{j!} u^j$$

is valid for every u. In the case where there are only finitely many distinct elements in ths sequence (which will happen in our application), $\rho(\alpha)$ will be

a step function making only finitely many steps. Then $L(u;\rho)$ will be a finite expression of the form $\sum_{j=1}^{r} c(\beta_j)e^{i\beta_r}$ and the values assumed are then $\beta_1, \cdots, \beta_r$, each with density $c(\beta_j)$.

As an illustration let us compute the asymptotic distribution for $\{\mu(n)\}_{n=1}^{\infty}$. By now the reader should have no difficulties in proving directly that this is merely the step function which has the saltus ('jump') $3/\pi^2,\ 1-6/\pi^2, 3/\pi^2$ at $\alpha = -1, 0, 1$, respectively. Let us derive this by the method of moments. The odd moments of μ are zero and the $2j$th moment with $j \geq 1$ equals $6/\pi^2$. The zeroth moment equals one. We thus derive that

$$L(u,\rho) = 1 - \frac{6}{\pi^2} + \frac{6}{\pi^2}\cos u = \frac{3}{\pi^2}e^{-iu} + 1 - \frac{6}{\pi^2} + \frac{3}{\pi^2}e^{iu},$$

from which we arrive at the same conclusion as before.

7.2 Computing the moments and asymptotic distribution function

Using the representation (7) for the Dirichlet series $f_n^{(2j)}(s)$ and $f_n^{(2j-1)}(s)$ and the Wiener Ikehara theorem, Theorem 1, the mean of the jth moment of $c_n(m)$ can be determined (note that by property 5 of Proposition 3 we can take $F(s) = m^j\zeta(s)$ in that result when we apply it to $f_n^{(j)}(s)$), We find that

$$M_n(c_n(m)^{2j}) = \frac{F_m^{(2j)}(1)}{\zeta(2)} = \frac{6}{\pi^2}\prod_{p|m} F^{(2j)}_{p^{\nu_p(m)}}(1). \tag{22}$$

Note that $M_n(c_n(m)^{2j})\zeta(2)$ is a multiplicative function in m. Similarly, we have $M_n(c_n(m)^{2j-1}) = 0$. (We use the subscript n in M to indicate that the mean is with respect to n.) Since $|c_n(m)|^{2j} \leq m^{2j}$ and hence the same inequality holds for the mean, it follows that the radius of convergence of the power series

$$G_m(u) = \sum_{j=0}^{\infty} \frac{M_n(c_n(m)^{2j})}{(2j)!}(-u^2)^j \tag{23}$$

is infinite for every m $(= 1, 2, \cdots)$. In case $m = p^r$, $G_m(u)$ is easily evaluated using (22) and (6). One obtains

$$G_{p^r}(u) = \left(1 - \frac{6}{\pi^2}\frac{1-p^{-k-2}}{1-p^{-2}}\right) + \frac{6}{\pi^2}\left(1+\frac{1}{p}\right)^{-1}\left(\sum_{j=0}^{r}\frac{\cos(\varphi(p^j)u)}{p^j} + \frac{\cos(p^r u)}{p^{r+1}}\right).$$

Note that if k is fixed, then

$$G_{p^r}(u) \to 1 - \frac{6}{\pi^2} + \frac{6}{\pi^2}\cos u,\ (p \to \infty) = G_1(u),$$

and

$$G_{p^r}(u) \to \left(1 - \frac{6}{\pi^2(1-p^{-2})}\right) + \frac{6}{\pi^2}(1+\frac{1}{p})^{-1}\sum_{j=0}^{r}\frac{\cos(\varphi(p^j)u)}{p^j},\ (k \to \infty).$$

For any given m it is not difficult to compute $G_m(u)$ using formula (22). The existence of all the moments $M_n(c_n(m)^j)$, $j = 0, 1, 2, \cdots$ implies the existence of an asymptotic distribution function ρ_m. Note that $L(u; \rho_m) = G_m(u)$ and that $G_m(u) = \sum_{j=1}^r c(\beta_j)e^{i\beta_r}$. The values assumed are $\beta_1, \cdots, \beta_r$, each with density $c(\beta_j)$. Thus, taking $m = p^r$ for example we immediately read off from the formula for $G_{p^r}(u)$ that

$$\delta(c_n(p^r) = v) = \begin{cases} 1 - \frac{6}{\pi^2}\frac{1-p^{-r-2}}{1-p^{-2}} & \text{if } v = 0; \\ \frac{3}{\pi^2 p^h(1+\frac{1}{p})} & \text{if } v = \pm\varphi(p^h), \text{ where } h = 1, \dots, r; \\ \frac{3}{\pi^2 p^{r+1}(1+\frac{1}{p})} & \text{if } v = \pm p^r. \end{cases}$$

8 Value distribution of cyclotomic coefficients

8.1 Evaluating cyclotomic coefficients

Write

$$\Phi_n(X) = \sum_{k=0}^{\varphi(n)} a_n(k)X^k.$$

We consider the value distribution of the cyclotomic coefficients $a_n(k)$ for fixed k, as n runs over all the positive integers. By setting $\mu(n/d) = 0$ whenever n/d is not an integer we have, for $n > 1$,

$$\Phi_n(X) = \prod_{d|n}(1 - X^d)^{\mu(n/d)} = \prod_{d=1}^{\infty}(1 - X^d)^{\mu(n/d)}. \tag{24}$$

Note that, for $|X| < 1$, we have

$$\prod_{d=1}^{\infty}(1 - X^d)^{\mu(n/d)} = \prod_{d=1}^{\infty}\left(1 - \mu(\frac{n}{d})X^d + \frac{1}{2}\mu(\frac{n}{d})(\mu(\frac{n}{d}) - 1)\sum_{j=1}^{\infty} X^{jd}\right), \tag{25}$$

where we used the observation that, for $|X| < 1$,

$$(1 - X^d)^{\mu(n/d)} = \left(1 - \mu(\frac{n}{d})X^d + \frac{1}{2}\mu(\frac{n}{d})(\mu(\frac{n}{d}) - 1)\sum_{j=1}^{\infty} X^{jd}\right). \tag{26}$$

From (25) it is not difficult to derive a formula for $a_n(k)$ for a fixed k; this is just the coefficient of X^k in the right hand side of (24) (this approach seems to be due to D.H. Lehmer [9]). We thus obtain, for $n > 1$,

$$\begin{cases} a_n(1) = -\mu(n); \\ a_n(2) = \mu(n)(\mu(n) - 1)/2 - \mu(n/2); \\ a_n(3) = \mu(n)^2/2 - \mu(n)/2 + \mu(n/2)\mu(n) - \mu(n/3). \end{cases}$$

More generally, we have

$$a_n(k) = \sum c(k_1, ..., k_s; e_1, \dots, e_s)\mu(\frac{n}{k_1})^{e_1} \cdots \mu(\frac{n}{k_s})^{e_s}, \tag{27}$$

where the sum is over all partitions $k_1 + \cdots + k_s$ of all the integers $\leq k$ with $k_1 \geq k_2 \geq \cdots k_s$ and over all $e_1, \dots, e_s$ with $1 \leq e_j \leq 2$ for $1 \leq j \leq s$. Using this result we will deduce that the n dependence of $a_n(k)$ is not that strong.

Proposition 7 *Put* $N_k = \mathrm{lcm}(1, 2, \cdots, k)\prod_{p\le k} p$. *We can uniquely decompose* n *as* $n = n_k c_k$ *with* $(c_k, N_k) = 1$ *and* n_k *and* c_k *natural numbers. There exist functions* A_1 *and* B_1 *with as domain the divisors of* N_k *such that*

$$a_n(k) = \begin{cases} A_1(n_k)\mu(c_k)^2 + B_1(n_k)\mu(c_k) & \text{if } n_k | N_k; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The assertion regarding the uniqueness of the decomposition $n = n_k c_k$ is trivial. For a given n only those partitions $k_1, k_2, \cdots, k_s$ contribute to (27) for which n/k_i is an integer for $1 \le i \le s$. Note that $k_i | n_k$. Thus, we can write

$$\mu(\frac{n}{k_1})^{e_1} \cdots \mu(\frac{n}{k_s})^{e_s} = \mu(\frac{n_k}{k_1})^{e_1} \cdots \mu(\frac{n_k}{k_s})^{e_s}\mu(c_k)^{e_1+\ldots+e_s}.$$

If $n_k \nmid N_k$, then none of the integers $n_k/k_1, ..., n_k/k_s$ is squarefree and $a_n(k) = 0$, so assume that $n_k|N_k$. On using that $\mu(r)^w$ with $w \ge 1$ either equals $\mu(r)$ or $\mu(r)^2$, the result follows from (27). □

The above proposition shows that $\{a_n(k)|n \in \mathbb{N}\}$ is a finite set, thus if we fix k, there are only finitely many possibilities for the values of the coefficient of X^k in a cyclotomic polynomial. We will now show that $-1, 0$ and 1 are always amongst these values. In the formula for $a_n(k)$ there is always the term $-\mu(n/k)$. Let us take $n = ck\prod_{p\le k} p$, where c only has prime divisor $> k$. Then all the terms of the form $\mu(n/r)$ with $1 \le r < k$ are zero (since either $r \nmid n$ or n/r is not squarefree) and we obtain that $a_n(k) = -\mu(c)(-1)^{\pi(k)}$, where $\pi(x)$ as usual denotes the number of primes $p \le x$ not exceeding x. In particular, it follows that $a_n(k)$ always assumes the values $-1, 0$ and 1. Combining this insight with the previous proposition we arrive at the following result.

Proposition 8 *Let* $k \ge 1$ *be fixed. Let* $N_k = \mathrm{lcm}(1, 2, \cdots, k)\prod_{p\le k} p$ *and let* $q > k$ *be any prime exceeding* k. *We have*

$$\{-1, 0, 1\} \subseteq \{0, a_d(k), a_{dq}(k) \mid d|N_k\} = \{a_n(k) \mid n \ge 1\}.$$

Using formula (10) it is seen that in the latter proposition N_k can be replaced by $k\prod_{p\le k} p$. We have $|a_n(k)| \le \max_{n\ge1} |a_n(k)| = B(k)$. See Table 2 for the values of $B(k)$ for $1 \le k \le 30$.

We next show that the inclusion in Proposition 8 is strict for $k \ge 13$. Our proof rests on the following rather elementary result on prime numbers.

Proposition 9 *For* $k \ge 13$ *there are consecutive odd primes* $p_1 < p_2 < p_3$ *such that* $p_3 \le k < p_1 + p_2$.

Proof. Breusch proved in 1934 that for $x \ge 48$ there is at least one prime in $[x, 9x/8]$ (this strengthens *Bertrand's Postulate* asserting that there is always a prime between x and $2x$, provided $x \ge 2$). Let $\alpha = 1.32$. A little computation shows that the above result implies that for $x \ge 9$ there is at least one prime in $[x, \alpha x]$. One checks that the assertion is true for $k \in [13, 21)$. Assume that $k \ge 21$ $(\ge 9\alpha^3)$. Let p_3 be the largest prime not exceeding k and let p_1 and p_2 be primes such that p_1, p_2 and p_3 are consecutive primes. Then $p_3 \ge k/\alpha$, $p_2 \ge k/\alpha^2$ and $p_1 \ge k/\alpha^3$. On noting that $p_1 + p_2 \ge k(1/\alpha + 1/\alpha^2) > k$, the proof is then completed. □

Proposition 10 *For $k \geq 13$ we have $\{-2,-1,0,1\} \in \{a_n(k) : n \in \mathbb{N}\}$ (and thus $B(k) \geq 2$).*

Proof. Let p_1, p_2 and p_3 be odd primes satisfying the condition of Proposition 9. Using (24) we infer that

$$\Psi_{p_1p_2p_3}(X) \equiv \frac{(1-X^{p_3})}{(1-X)}(1-X^{p_1})(1-X^{p_2}) \equiv (1+X+\cdots+X^{p_3-1})(1-X^{p_1}-X^{p_2}),$$

where we computed modulo X^{k+1}. It follows that $a_{p_1p_2p_3}(k) = -2$. The proof is completed on invoking Proposition 8. □

8.1.1 Numerical evaluation of $a_n(k)$ for small k

For our purposes it is relevant to be able to numerically evaluate $a_n(k)$ for small k and large n. A computer package like Maple evaluates $a_n(k)$ be evaluating the whole polynomial $\Phi_n(x)$. For large n this is far too costly. Instead it is more efficient to use (24) and expand for every d for which $\mu(n/d) \neq 0$, $(1-X^d)^{\mu(n/d)}$ as a Taylor series up to $O(x^{k+1})$ and multiply all these series together. The most efficient method to date is due to Grytczuk and Tropak [7]. First they apply formula (10). Thus it is enough to compute $a_n(k)$ with n squarefree. If $\phi(n) < k$, then $a_n(k) = 0$, so we may assume that $\phi(n) \geq k$. Let $d = (n, \prod_{p \leq k} p)$. Put $T_r = \mu(n)\mu((r,d))\varphi((r,d))$. Compute $b_0, \ldots, b_k$ recursively by $b_0 = 1$ and

$$b_j = -\frac{1}{j}\sum_{m=0}^{j-1} b_m T_{j-m} \text{ for } j > 0.$$

Then $b_k = a_n(k)$. The proof uses the formula

$$a_n(k) = -\frac{1}{k}\sum_{m=0}^{k-1} a_n(m) c_n(k-m) \text{ for } k > 1, \tag{28}$$

which follows by Viete's and Newton's formulae from (8) and it uses Property 2 of Proposition 3. An alternative proof of (28) is obtained on using the following observation together with Property 1 of Proposition 3.

Proposition 11 *Suppose that, as formal power series,*

$$\prod_{d=1}^{\infty}(1-X^d)^{-a_d} = \sum_{d=0}^{\infty} r(d)X^d,$$

then $dr(d) = \sum_{j=1}^{d} r(d-j)\sum_{k|j} ka_k$.

Proof. Taking the logarithmic derivative of $\prod_{d=1}^{\infty}(1-X^d)^{-a_d}$ we obtain

$$\frac{\sum_{d=1}^{\infty} dr(d)X^d}{\sum_{d=0}^{\infty} r(d)X^d} = X\frac{d}{dx}\log\prod_{d=1}^{\infty}(1-X^d)^{-a_d} = \sum_{j=1}^{\infty}(\sum_{k|j} ka_k)X^j,$$

whence the result follows. $\square$

For every integer v it is a consequence of Proposition 5 that there exists a minimal integer k, $k_{\min}$, such that there exists a natural number n with $a_n(k_{\min}) = v$. Grytczuk and Tropak [7, Table 2.1] used their method to determine $k_{\min}$ for the integers in the interval $[-9, \ldots, 10]$. Gallot has extended this range from $[-60, \ldots, 70]$. From Table 11 one can determine $k_{\min}$ for the range $[-15, \ldots, 15]$.

8.2 Möller's paper reconsidered

In this section we reconsider Möller's paper [13]. In Möller's approach $a_n(k)$ is connected with partitions of k. A partition of a positive integer m is an expression of the form $m_1+m_2+\cdots+m_r = m$ with all the $m_j \geq 0$. Ordering is disregarded. Thus, $1+1+2+3$ and $3+2+1+1$ are considered to be the same partitions of 7. A partition can be identified with a sequence $\{n_j\}_{j=0}^{\infty}$ of non-negative integers satisfying $\sum_j jn_j = m$. W.l.o.g. we can denote a partition, λ, of k as $\lambda = (k_1^{n_{k_1}} \cdots k_s^{n_{k_s}})$, where $n_{k_1} \geq n_{k_2} \geq \ldots \geq n_{k_s} \geq 1$ (thus the number k_j occurs n_{k_j} times in the partition). The set of all partitions of m will be denoted by $\mathcal{P}(m)$. The number of different partitions of m is denoted by $p(m)$. Hardy and Littlewood in 1918, and Uspensky independently in 1920, proved that

$$p(m) \sim e^{\pi\sqrt{2m/3}}/(4m\sqrt{3}) \text{ as } m \to \infty.$$

Proposition 12 [13, Satz 2]. *We have*

$$a_n(k) = \sum_{\sum_j jn_j = k,\ n_j \geq 0} \prod (-)^{n_j} \binom{\mu(\frac{n}{j})}{n_j}.$$

Proof. The taylor series of $(1-X)^a$ equals, for $|X| < 1$, $\sum_{j=0}^{\infty}(-1)^j\binom{a}{j}X^j$, where $\binom{a}{j} = a(a-1)\cdots(a-(j-1))/j!$. Using this we infer that

$$(1-X^d)^{\mu(\frac{n}{d})} = \sum_{j=0}^{\infty}(-1)^j \binom{\mu(\frac{n}{d})}{j} X^{Dj},\ |X| < 1, \tag{29}$$

The proof now follows from (29) and (24). $\square$

Our proof above shows that the formula given in Proposition 12 is a triviality, whereas Möller's ingenious and rather involved proof of it obscures this.

Möller uses Proposition 12 to show that

$$M_n(a_n(k)) = \lim_{x\to\infty} \frac{\sum_{n\leq x} a_n(k)}{x}$$

exists and gives a formula for it. To do so he first determines the average of $\prod(-1)^{n_j}\binom{\mu(n/j)}{n_j}$. He does this by expanding it out as a sum of terms of the form $\prod \mu^{\alpha_j}(n/j)$ with $1 \leq \alpha_j \leq 2$, which is possible by (30). The average of each

term $\prod \mu^{\alpha_j}(n/j)$ is easily determined. He then invokes Proposition 12 and uses combinatorics to simplify his expressions. His final result is still quite complicated and we did not feel inclined to check its equivalence with our more simple formula in Theorem 3 below. We will now follow Möller's trail, in spirit, but not in detail and see where it leads us.

Proposition 13 *Let $s \geq 1$, $k_1, \cdots, k_s$ be distinct integers and $n_{k_1} \geq n_{k_2} \geq \ldots \geq n_{k_s} \geq 1$. If $n_{k_1} \geq 2$ we let t be the largest integer $\leq s$ for which $n_{k_t} \geq 2$, otherwise we let $t = 0$. Let $L = [k_1, \ldots, k_s]$ and $G = (k_1, \ldots, k_s)$. We have*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n \leq x} (-1)^{n_{k_1}+\cdots+n_{k_s}} \binom{\mu(\frac{n}{k_1})}{n_{k_1}} \cdots \binom{\mu(\frac{n}{k_s})}{n_{k_s}} = \frac{6}{\pi^2} \frac{\epsilon \mu(\frac{L}{k_{t+1}}) \cdots \mu(\frac{L}{k_s})}{G \prod_{p|\frac{L}{G}} (p+1)},$$

where

$$\epsilon = \begin{cases} 1 & \text{if } n_{k_1} = 1,\ s \text{ is even and } \mu(L/G) \neq 0; \\ \mu(\frac{L}{k_1})^{s-t}/2 & \text{if } n_{k_1} \geq 2 \text{ and } \mu(L/k_1) = \cdots = \mu(L/k_t) \text{ and } \mu(L/G) \neq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 1 *We have*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n \leq x} (-1)^{n_{k_1}+\cdots+n_{k_s}} \binom{\mu(\frac{n}{k_1})}{n_{k_1}} \cdots \binom{\mu(\frac{n}{k_1})}{n_{k_1}} =$$

$$\frac{3}{\pi^2 G \prod_{p|\frac{L}{G}} (p+1)} \left(\prod_{j=1}^{s} (-1)^{n_{k_j}} \binom{\mu(\frac{L}{k_j})}{n_{k_j}} + \prod_{j=1}^{s} (-1)^{n_{k_j}} \binom{-\mu(\frac{L}{k_j})}{n_{k_j}} \right).$$

Proof of Proposition 13. Put

$$S(x) = \sum_{n \leq x} (-1)^{n_{k_1}+\cdots+n_{k_s}} \binom{\mu(\frac{n}{k_1})}{n_{k_1}} \cdots \binom{\mu(\frac{n}{k_s})}{n_{k_s}}.$$

Comparison of (26) and (29) yields

$$(-1)^j \binom{\mu(\frac{n}{d})}{j} = \begin{cases} 1 & \text{if } j = 0; \\ -\mu(n/d) & \text{if } j = 1; \\ \mu(n/d)(\mu(n/d)-1)/2 & \text{if } j \geq 2. \end{cases} \tag{30}$$

Note that for $j \geq 2$ the binomial coefficient is only non-zero if $\mu(n/d) = -1$. Using (30) it follows that a necessery condition for the argument of $S(x)$ to be non-zero is that $L|n$. Now write $n = mL$. Note that $\mu(mL/k_1) \cdots \mu(mL/k_s) = \mu(m)^s \mu(L/k_1) \cdots \mu(L/k_s)$ if $(m, L/k_j) = 1$ for $1 \leq j \leq s$ and equals zero otherwise. It is not difficult to show that $[\frac{L}{k_1}, \ldots, \frac{L}{k_s}] = \frac{L}{G}$ and using this, that $\mu(L/k_1), \cdots, \mu(L/k_s)$ are squarefree iff L/G is squarefree. It follows that if $\mu(L/G) = 0$, then $S(x) = 0$ and we are done, so next assume that $\mu(L/G) \neq 0$. We infer that

$$S(x) = \sum_{m \leq x/L,\ (m, L/G)=1} (-1)^{n_{k_1}+\cdots+n_{k_s}} \binom{\mu(mL/k_1)}{n_{k_1}} \cdots \binom{\mu(mL/k_s)}{n_{k_s}}.$$

Let us first consider the (easy) case where $n_{k_1} \geq 1$. Then we obtain $S(x) = (-1)^s \mu(\frac{L}{k_1}) \cdots \mu(\frac{L}{k_s}) \sum_{m \leq x/L,\ (m,L/G)=1} \mu(m)^s$. If s is odd, then by Lemma 2 it follows that $\lim_{x\to\infty} S(x)/x = 0$ and we are done, so next assume that s is even. Then we apply Lemma 1 and obtain that

$$\lim_{x\to\infty} \frac{S(x)}{x} = \frac{6\mu(\frac{L}{k_1}) \cdots \mu(\frac{L}{k_s})}{\pi^2 L \prod_{p|\frac{L}{G}} (1+\frac{1}{p})}.$$

The assumption $\mu(L/G) \neq 0$ implies that $L\prod_{p|L/G}(1+1/p) = G\prod_{p|L/G}(p+1)$. Next we consider the case where $n_{k_1} \geq 1$. The corresponding binomial coefficient is only non-zero if $\mu(mL/k_1) = -1$. Similarly, we must have $\mu(mL/k_j) = -1$ for $1 \leq j \leq t$. It follows that if it is not true that $\mu(L/k_1) = \ldots = \mu(L/k_t)$, then $S(x) = 0$ and hence $\lim_{x\to\infty} S(x)/x = 0$ as asserted, so assume that $\mu(L/k_1) = \ldots = \mu(L/k_t)$. We have

$$\begin{aligned} S(x) &= \sum_{\substack{m \leq x/L,\ (m,L/G)=1 \\ \mu(mL/k_1)=-1}} (-\mu(m))^{s-t} \mu(\frac{L}{k_1}) \cdots \mu(\frac{L}{k_1}) \\ &= (\mu(\frac{L}{k_1}))^{s-t} \mu(\frac{L}{k_{t+1}}) \cdots \mu(\frac{L}{k_s}) \sum_{\substack{m \leq x/L,\ (m,L/G)=1 \\ \mu(m)=-\mu(L/k_1)}} 1. \end{aligned}$$

On invoking Lemma 1 and Lemma 2 the proof is then completed. □

Proof of Corollary 1. Follows from a case by case analysis from Proposition 13 on using (30). □

Theorem 3 *We have*

$$M_n(a_n(k)) = \frac{3}{\pi^2} \sum_{\substack{\lambda=(k_1^{n_{k_1}} \ldots k_s^{n_{k_s}}) \in \mathcal{P}(k) \\ n_{k_1} \geq \ldots n_{k_s} \geq 1}} \frac{\epsilon(\lambda)}{G(\lambda) \prod_{p|\frac{L(\lambda)}{G(\lambda)}} (p+1)},$$

where

$$\epsilon(\lambda) = \prod_{j=1}^{s} (-1)^{n_{k_j}} \binom{\mu(\frac{L}{k_j})}{n_{k_j}} + \prod_{j=1}^{s} (-1)^{n_{k_j}} \binom{-\mu(\frac{L}{k_j})}{n_{k_j}},$$

$L(\lambda) = [k_1, \ldots, k_s]$ *and* $G(\lambda) = (k_1, \ldots, k_s)$.

Proof. The result follows from Proposition 12 together with Corollary 1. □

In case k is a prime, $G(\lambda) = 1$ for every partition and the above formula further simplifies. The above formula suggests a connection with the group or representation theory of the symmetric group S_k. The conjugacy classes in S_k are in 1-1 correspondence with the partitions of k. If $\lambda = (k_1^{n_{k_1}} \ldots k_s^{n_{k_s}}$, then the order of every element in the corresponding conjugacy class equals $L(\lambda)$. In particular, $L(\lambda) \leq g(n)$, where $g(n)$ denotes the maximum of all orders of elements in S_k. It was shown by E. Landau in 1903 that $\log g(k) \sim \sqrt{k \log k}$ as k tends to infinity (for a nice account of this see [11]), whereas by Stirling's theorem $\log k! \sim k \log k$.

From Proposition 12 Möller infers that $|a_n(k)| \le p(k) - p(k-2)$. To see this note that the partitions having 1 occurring at least twice do not contribute if $\mu(n) \in \{0, 1\}$. If $\mu(n) = -1$, then either $\mu(2n) = 1$ or $\mu(n/2) = 1$. This in combination with (11) allows us then to argue as before and leads us to the same bound. By a much more involved argument, using some analytic number theory, Möller concludes that $B(k) > k^m$ for $k \ge k_0(m)$. This result was sharpened by several later authors culminating in Bachman's estimate (12).

Möller used Proposition 12 to evaluate $B(k)$ for $1 \le k \le 30$. He did this by hand. The outcome is in Table 2. We redid this computation by computer and arrived the same result. Note that $M_n(a_n(k)) = 6e_k/\pi^2$, with e_k a rational number. For $1 \le k \le 20$ we give the value of e_k in Table 3 (our table agrees with the one given in [13], except for the incorrect values $e_{10} = 319/1440$ and $e_{16} = 733/2016$ appearing there).

Table 3: Scaled average, $e_k = \zeta(2)M_n(a_n(k))$, of $a_n(k)$

j	1	2	3	4	5	6	7	8	9	10
e_j	0	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{8}$	$\frac{7}{24}$	$\frac{1}{18}$	$\frac{7}{24}$	$\frac{19}{144}$	$\frac{31}{160}$
j	11	12	13	14	15	16	17	18	19	20
e_j	$\frac{1}{16}$	$\frac{55}{192}$	$\frac{13}{288}$	$\frac{61}{288}$	$\frac{2287}{20160}$	$\frac{733}{4032}$	$\frac{667}{8064}$	$\frac{79}{336}$	$\frac{55}{1344}$	$\frac{221}{960}$

Regarding e_k Möller proposed:

Conjecture 2 [13]. *Let $k \ge 1$. Write $M_n(a_n(k)) = 6e_k/\pi^2$.*
1) *We have $0 \le e_k \le 1/2$.*
2) *We have $(-1)^k(e_k - e_{k+1}) > 0$.*

Möller stated that with help of an IBM 7090 he wanted to check his conjecture for further values of k. Had he carried this out, he would have discovered that $(-1)^{34}(e_{34} - e_{35}) = -18059/4626720 < 0$. Other counterexamples occur at $k = 35, 45$ and 94. Indeed, we would not be surprised if part 2 of the Conjecture is violated for infinitely many k.

On the other hand, part 1 of the Conjecture is true for $k \le 75$. The numbers e_k seem to be decreasing to zero and their size seems to be related to the number of prime factors of k, the more prime factors the larger e_k seems to be. In Table 11 (kindly computed by Yves Gallot) we give values for e_k for k up to 61.

8.3 Average and value distribution

We give, using Proposition 7, a simpler formula for $M_n(a_n(k))$ involving $a_n(k)$ for a finite set of n.

Theorem 4 *Let $k \ge 1$ be fixed. Put $M_k = k\prod_{p \le k} p$, and let $q > k$ be any prime. Then*
$$M_n(a_n(k)) = \frac{3}{\pi^2 \prod_{p \le k}(1 + \frac{1}{p})} \sum_{d | M_k} \frac{a_d(k) + a_{dq}(k)}{d}.$$

Furthermore, when $v \neq 0$,

$$\delta(a_n(k) = v) = \frac{3}{\pi^2 \prod_{p \le k}(1 + \frac{1}{p})} (\sum_{\substack{d|M_k \\ a_d(k)=v}} \frac{1}{d} + \sum_{\substack{d|M_k \\ a_{dq}(k)=v}} \frac{1}{d}).$$

Proof. Let $N_k = \mathrm{lcm}(1, 2, \cdots, k) \prod_{p \le k} p$, $r_1 = \prod_{p \le k} p$ and $r_2 = r_1 N_k$. We have

$$\begin{aligned}
\sum_{n \le x} a_n(k) &= \sum_{d|N_k} \sum_{\substack{n \le x \\ (n, r_2)=d}} (A_1(d)\mu(\frac{n}{d})^2 + B_1(d)\mu(\frac{n}{d})) \\
&= \sum_{d|N_k} \sum_{\substack{m \le x/d \\ (m, r_2/d)=1}} (A_1(d)\mu(m)^2 + B_1(d)\mu(m)) \\
&= \sum_{d|N_k} \sum_{\substack{m \le x/d \\ (m, r_1)=1}} (A_1(d)\mu(m)^2 + B_1(d)\mu(m)) \\
&= \sum_{d|N_k} A_1(d) \sum_{\substack{m \le x/d \\ (m, r_1)=1}} \mu(m)^2 + o_k(x),
\end{aligned}$$

where we used Proposition 7 and Lemma 2. On invoking Lemma 1 we then obtain that

$$\sum_{n \le x} a_n(k) = \frac{6}{\pi^2 \prod_{p \le k}(1 + \frac{1}{p})} \sum_{d|N_k} \frac{A_1(d)}{d} + o_k(x).$$

On noting that $A_1(d) = (a_d(k) + a_{dq}(k))/2$ (and $B_1(d) = (a_d(k) - a_{dq}(k))/2$, but this is not needed), the first formula follows, but with N_k instead of M_k. On invoking formula (10) it is easily seen that

$$\sum_{d|N_k} \frac{a_d(k) + a_{dq}(k)}{k} = \sum_{d|M_k} \frac{a_d(k) + a_{dq}(k)}{k}$$

and hence the first formula follows. The proof of the second formula is similar and left to the reader. □

Using identity (11) we arrive at the following corollary to this theorem:

Corollary 2 *Let $k \ge 3$ be fixed and odd and M_k and q as in Theorem 4. Then*

$$M_n(a_n(k)) = \frac{1}{\pi^2 \prod_{2 < p \le k}(1 + \frac{1}{p})} \sum_{d|M_k/2} \frac{a_d(k) + a_{dq}(k)}{d}.$$

Furthermore, when $v \neq 0$,

$$\delta(a_n(k) = v) = \frac{3}{\pi^2 \prod_{p \le k}(1 + \frac{1}{p})} (\sum_{\substack{d|M_k/2 \\ a_d(k)=v}} \frac{1}{d} + \sum_{\substack{d|M_k/2 \\ a_d(k)=-v}} \frac{1}{2d} + \sum_{\substack{d|M_k/2 \\ a_{dq}(k)=v}} \frac{1}{d} + \sum_{\substack{d|M_k/2 \\ a_{dq}(k)=-v}} \frac{1}{2d}).$$

As a corollary of Theorem 4 and Corollary 2 we have:

Proposition 14 *We have $e_k 2k \prod_{p \le k}(p + 1) \in \mathbb{Z}$. In case k is odd we have $e_k k \prod_{p \le k}(p + 1) \in \mathbb{Z}$.*

Yves Gallot observed that actually for $k \le 100$, we have $e_k k \prod_{p \le k}(p+1) \in \mathbb{Z}$.

In case k is prime, the divisor sum in the previous corollary can be further reduced:

Proposition 15 *Let $k \ge 3$ be a fixed prime. Put $N_k = \prod_{2<p<k} p$, and let $q > k$ be any prime. Then*

$$M_n(a_n(k)) = \frac{1}{\pi^2 \prod_{2<p<k}(1+\frac{1}{p})} \sum_{d|N_k} \frac{a_d(k) + a_{dq}(k)}{d}.$$

Proof. We consider the formula given in the previous corollary. The divisors of $M_k/2$ are either of the form d with $d|kR_k$ or of the form dk^2 with $d|R_k$. For the latter d we find, using (10) that $a_{dk^2}(k) = a_{dk}(1) = -\mu(dk)$ and hence $\sum_{d|M_k/2}(a_d(k)+a_{dq}(k))/d = \sum_{d|kR_k}(a_d(k)+a_{dq}(k))/d$. Now suppose that $d|R_k$. Using Proposition 12 we infer that

$$a_{dk}(k) + a_{dkq}(k) = a_d(k) + a_{dq}(k) + \mu(d) + \mu(dq) = a_d(k) + a_{dq}(k).$$

Using this observation it follows that

$$\sum_{d|kR_k} \frac{a_d(k) + a_{dq}(k)}{d} = (1+\frac{1}{k}) \sum_{d|R_k} \frac{a_d(k) + a_{dq}(k)}{d},$$

whence the result follows. □

Clearly $\delta(a_n(k) = 0) = 1 - \sum_{v \ne 0} \delta(a_n(k) = v)$, where the latter sum has only finitely many non-zero values and it is a finite computation to determine those v for which $a_n(k) = v$ for some n. For $1 \le k \le 16$ the non-zero values of $\zeta(2)\delta(a_n(k) = v)$ are given in Table 4 (except for $v = 0$). A more extensive version of Table 4 is provided by Table 11.

Table 4: Value of $\zeta(2)\delta(a_n(k) = v)$

	$v=-2$	$v=-1$	$v=1$	$v=2$
$k=1$	0	1/2	1/2	0
$k=2$	0	1/12	7/12	0
$k=3$	0	5/24	3/8	0
$k=4$	0	1/6	1/2	0
$k=5$	0	13/80	23/80	0
$k=6$	0	25/144	67/144	0
$k=7$	1/576	577/2688	731/2688	1/1152
$k=8$	0	1/8	5/12	0
$k=9$	0	65/384	347/1152	0
$k=10$	0	161/960	347/960	0
$k=11$	1/2304	8299/50688	11489/50688	1/4608
$k=12$	0	349/2304	1009/2304	0
$k=13$	43/48384	219269/1257984	277171/1257984	43/96768
$k=14$	13/21504	2395/21504	2319/7168	1/2304
$k=15$	13/32256	1345/7168	97247/322560	13/64512
$k=16$	5/21504	12149/64512	1127/3072	5/2688

Let us now look at Theorem 3 and Theorem 4 from the viewpoint of computational complexity. In Theorem 3 the sum has $p(k)$ terms and the famous estimate of Hardy and Littlewood yields that $\log p(k) \sim \pi\sqrt{2k/3}$ as k tends to infinity. In Theorem 4 we sum over $t(k)$ terms where $\log t(k) \sim \pi(k)\log 2 \sim k\log 2/\log k$. So Theorem 3 yields the computational superior method. Theorem 4 is, however, much more easily implemented. Using Proposition 12 in combination with Theorem 4 a result comparable with Theorem 3 is obtained. Indeed, if one starts with Proposition 15 and invokes Proposition 12, one obtains a sum over partitions of k, where now 2 for example does not occur in the partition. This yields a result superior in complexity to that provided by Theorem 3.

As already pointed out by Möller one could use his method to study the value distribution of $a_n(k)$ in case e.g. $B(k) = 2$ by considering the integer $a_n(k)(a_n(k)-1)/2$ to determine $\delta(a_n(k) = -1)$ for example. This then yields a sum with $p(k)^2$ terms and this results in an algorithm that has worse complexity than that provided by Theorem 4. Aside from this, this seems to be, from the practical point of view, an unwieldy method.

8.3.1 Some observations related to Table 4

In this section we make some observations regarding Table 4 and Table 11, which is a much more extensive version of Table 4.

Let us put $\mathcal{B}(k) = \{a_n(k) : n \in \mathbb{N}\}$. Recall that $|\mathcal{B}(k)| = B(k)$. Recall also that $-1, 0, 1 \in \mathcal{B}(k)$. Using this and Corollary 2 we infer that if k is odd, then $\mathcal{B}(k)$ is symmetric: we have $v \in \mathcal{B}(k)$ iff $-v \in \mathcal{B}(k)$. For k is even numerical results suggest that often $\mathcal{B}(k)$ is not symmetric. For $k \le 75$ it is true that if $v \in \mathcal{B}(k)$ and v is negative, then $-v \in \mathcal{B}(k)$. An other observation that can be made is that for $k \le 75$ it is true that $\mathcal{B}(k)$ consists of consecutive integers, i.e. if $v_0 < v_1$ are in $\mathcal{B}(k)$, then so are all integers between v_0 and v_1.

Let us define $\mathcal{B}_j(k) = \{a_n(k) : n \equiv j (\mathrm{mod}\ 2)\}$, for $1 \le j \le 2$. It is easy to see that always $0 \in \mathcal{B}_j(k)$. Note that for k is even we have $\mathcal{B}_0(k) \subseteq \mathcal{B}_1(k) = \mathcal{B}(k)$. For k is odd we have $v \in \mathcal{B}_1(k)$ iff $-v \in \mathcal{B}_0(k)$. We could also express this as $\mathcal{B}_0(k) = \{-v : v \in B_1(k)\}$. Thus in this case, if we know $\mathcal{B}_0(k)$, we also know $\mathcal{B}_1(k)$. Inspection of Table 11 shows that for odd integers k with $B(k) \ge 2$ often $\delta(a_n(k) = B(k))$ and $\delta(a_n(k) = -B(k))$ differ by a factor two. Regarding this situation we have the following result:

Proposition 16 *Let k be odd and $v \neq 0$. We have*

$$2\delta(a_n(k) = v) = \delta(a_n(k) = -v) \text{ iff } v \notin \mathcal{B}_1(k) \text{ (that is iff } -v \notin \mathcal{B}_0(k)).$$

Proof. '$\Rightarrow$' An easy consequence of Corollary 2. '$\Leftarrow$' This uses in addition that $\{a_n(k) : 2 \nmid n\} = \{a_d(k) : d | M_k/2\}$. □

Example. Inspection of Table 4 shows that the condition of the proposition is satisfied for $k = 7$ and $v = 2$. It thus follows that there is no even integer n for which $a_n(7) = -2$ (whereas $a_{105}(7) = -2$). Further examples can be derived from Table 5.

Table 5: Set theoretic difference $\mathcal{B}(k)\backslash\mathcal{B}_0(k)$

$k=7$	$\{-2\}$	$k=11$	$\{-2\}$
$k=13$	$\{-2\}$	$k=15$	$\{-2\}$
$k=17$	$\{-3\}$	$k=19$	$\{-3\}$
$k=21$	$\{-3\}$	$k=23$	$\{-4,-3\}$
$k=25$	$\{-3\}$	$k=31$	$\{-4\}$
$k=35$	$\{5\}$	$k=37$	$\{5\}$
$k=39$	$\{5,6\}$	$k=43$	$\{-7\}$
$k=45$	$\{-7\}$	$k=47$	$\{-9,-8\}$
$k=51$	$\{8\}$	$k=53$	$\{9,10,11,12,13\}$

For Proposition 16 to be of some mathematical value we would hope that infinitely often $\mathcal{B}_0(k)$ is strictly contained in $\mathcal{B}(k)$. The next result shows that $a_{2n}(k)$ assumes all values as n and k range over the integers.

Proposition 17 *We have* $\{a_{2n}(k) : n,k \in \mathbb{N}\} = \mathbb{Z}$.

Proof. Given any integer $s \geq 2$ it is a consequence of the prime number theorem that it is possible to find primes $2 < p_1 < p_1+2 < p_2 < \cdots < p_s \leq p_1+p_2-2$. Let $k = p_1+p_2$ and let q be any prime $> k$. If s is even, let $m = 2p_1\cdots p_s q$, otherwise let $m = 2p_1\cdots p_s$. We claim that

$$\Phi_m(X) = \sum_{j=0}^{k-3} a_m(j)X^j + (s-1)X^{k-2} - (s-1)X^{k-1} \pmod{X^k}. \tag{31}$$

The claim together with $a_4(1)=0$ yields the result. We next prove (31). Using the observation that $\{d : d|m,\ d<k\} = \{1,2,p_1,\cdots,p_s,2p_1\}$ we infer from (24) that mod X^k we have

$$\begin{aligned}
\Phi_m(X) &\equiv \frac{(1-X)(1-X^{2p_1})}{(1-X^2)(1-X^{p_1})\cdots(1-X^{p_s})} \pmod{X^k} \\
&\equiv \frac{1+X^{p_1}}{(1+X)(1-X^{p_2})\cdots(1-X^{p_s})} \pmod{X^k} \\
&\equiv (1-X+X^2-X^3\cdots)(1+X^{p_1}+\cdots+X^{p_s}) \pmod{X^k} \\
&\equiv \sum_{j=0}^{k-3} a_m(j)X^j + (s-1)X^{k-2} - (s-1)X^{k-1} \pmod{X^k}.
\end{aligned}$$

This concludes the proof. □

The above argument shows for example that

$$\Phi_{2\cdot19\cdot23\cdot29\cdot31\cdot37\cdot39}(X) = \cdots - 5X^{40} + 5X^{41} + \cdots$$

9 Value distribution of $s_m(p)$ and $S_m(p)$

9.1 Ramanujan sums, cyclotomic polynomials and primitive roots

Proposition 19 connects the mod p reductions of totally symmetric functions in primitive roots to totally symmetric functions in primitive roots of unity. Our proof rests on the following simple result.

Proposition 18 *Let p be a prime not dividing m. Then $p|\Phi_m(a)$ iff the order of $a(\text{mod } p)$ is m.*

Proof. If the order of $a(\text{mod } p)$ equals m, then $a^d \not\equiv 1(\text{mod } p)$ for every $d < m$ and $a^m \equiv 1(\text{mod } p)$. By (9) we then infer that $p|\Phi_m(a)$. Since there are $\varphi(m)$ elements of order m in $(\mathbb{Z}/p\mathbb{Z})^*$, the equation $\Phi_m(X) \equiv 0(\text{mod } p)$ has at least $\varphi(m)$ distinct solutions mod p, however since the degree of $\Phi_m(X)$ equals $\varphi(m)$, these are all the solutions. □

Remark. We think this proof is more enlighting than that given in [16, p. 209]. A related, less well-known result is that $a(a^{p-1}-1)\Psi'_{p-1}(a)/\Psi_{p-1}(a)$ is congruent to $-1(\text{mod } p)$ if a is a primitive root mod p and divisible by p otherwise [18].

Proposition 19 *Let p be a prime. Put $t = \varphi(p-1)$. Let $g_1, \cdots, g_t$ be the modulu p distinct primitive roots. Let $1 \le j_1 < j_2 < \cdots < j_t \le p-1$ be the natural numbers coprime to $p-1$. Let $f(y_1, \cdots, y_t)$ be any totally symmetric function in the variables $y_1, \cdots, y_t$. Then*

$$f(g_1, \cdots, g_t) \equiv f(\zeta_{p-1}^{j_1}, \cdots, \zeta_{p-1}^{j_t}) \text{ (mod } p).$$

Proof. By Newton's result on totally symmetric polynomial functions it is sufficient to prove the result for all elementary totally symmetric polynomial functions. By Proposition 18 we infer that

$$\Phi_{p-1}(x) \equiv (x-g_1)\cdots(x-g_t) \text{ (mod } p). \tag{32}$$

On the other hand we have

$$\Phi_{p-1}(x) = (x-\zeta_{p-1}^{j_1})\cdots(x-\zeta_{p-1}^{j_t}). \tag{33}$$

From (32) and (33), the result easily follows. □

Let $g_1, \cdots, g_{\varphi(p-1)}$ denote the distinct primitive roots mod p with $1 \le g_j \le p-1$. Recall that

$$S_k(p) = \sum_{i=1}^{\varphi(p-1)} g_i^k,$$

and that $s_k(p)$ is the kth elementary totally symmetric function in $g_1, \cdots, g_{\varphi(p-1)}$. By Proposition 19 we infer that

$$S_k(p) \equiv c_{p-1}(k) \text{ (mod } p).$$

The latter congruence together with part 2 of Proposition 3 then yields the following lemma.

Lemma 4 *Let p be a prime. Then*

$$S_k(p) \equiv \mu\left(\frac{p-1}{(p-1,k)}\right)\frac{\varphi(p-1)}{\varphi(\frac{p-1}{(p-1,k)})} \pmod p.$$

This result is due to Moller [14], who proved it in a different (and more roundabout) way. Note that in case $k=1$ we have $S_1(p) \equiv \mu(p-1) \pmod p$, a result first established by Gauss [6].

If $\varphi(p-1) < k$, then $s_k(p) = 0$. If $k = \varphi(p-1)$, then we have, by Proposition 19,

$$s_k(p) = g_1 g_2 \cdots g_{\varphi(p-1)} \equiv (-1)^{\varphi(p-1)}\Phi_{p-1}(0) \equiv 1 (\mathrm{mod}\ p).$$

If $\varphi(p-1) > k$, then $s_k(p) \equiv (-1)^k a_{p-1}(\varphi(p-1)-k) = (-1)^k a_{p-1}(k) (\mathrm{mod}\ p)$. Taking the results of these three cases together we infer that

$$s_k(p) \equiv (-1)^k a_{p-1}(k) (\mathrm{mod}\ p).$$

The latter congruence and the congruence $S_k(p) \equiv c_{p-1}(k) \pmod p$ are the starting point of our analysis of their value distribution which is carried out in the next two sections.

9.2 Value distribution of $S_k(p)$ mod p

As we have seen $S_k(p) \equiv c_{p-1}(k) \pmod p$ and so instead of the value distribution of $S_k(p)$ mod p we can investigate the value distribution of $c_{p-1}(k)$. Given a natural number k, let $D_k = \{q_1, \ldots, q_t\}$ denote the set of prime divisors of k, ordered in size. We have $t = \#D_k = \omega(k) = \sum_{p|k} 1$. The following observation will play an important role.

Proposition 20 *On each prime p with $\nu_{D_k}(p-1)$ prescribed and $\mu_{D_k}(p-1)$ (respectively $\mu_{D_k}(p-1)^2$) prescribed, $c_{p-1}(k)$ (respectively $|c_{p-1}(k)|$) assumes the same value. If $\mu_{D_k}(p-1) = 0$, this value is zero.*

Our most general result regarding the value distribution of $c_{p-1}(k)$ reads as follows.

Theorem 5 *Let $v \neq 0$.*
1) *For every $H > 0$ we have*

$$\sum_{\substack{p \le x \\ |c_{p-1}(k)| = v}} 1 = c_v A \mathrm{Li}(x) + O\left(\frac{x}{\log^H x}\right),$$

where the O-constant depends at most on k and H and c_v is a rational number.
2) *If we assume in addition Conjecture 1, then the density of primes p with $c_{p-1}(k) = v$ exists and equals $c_v A/2$.*

Proof. Part 1. Write $\nu_{D_k}(p-1) = (e_1, \ldots, e_{\omega(k)})$. If $e_j \ge \nu_{q_j}(k) + 2$ for some $1 \le j \le \omega(k)$, then $\mu((p-1)/(p-1,k)) = 0$ and hence $c_{p-1}(k) = 0$. It follows there are only finitely many possibilities for $\nu_{D_k}(p-1)$ to be considered. To each

of them, by Proposition 20, we can associate an unique non-zero value of $c_{p-1}(k)$. We then have

$$\sum_{\substack{p\le x\\ |c_{p-1}(k)|=v}} 1 = \sum_{\substack{p\le x\\ \nu_{D_k}(p-1)\in T}} \mu_{D_k}(p-1)^2,$$

for some effectively computable set T. The result then follows by Proposition 6.
Part 2. The proof of part 2 is similar to that of part 1 and left to the reader. □

Conjecture 1 together with Theorem 2 (with $r = 1$ and $k = 2$) and the observation that $c_{p-1}(1) = \mu(p-1)$, then yields the following result demonstrating part 2 of Theorem 5 (with $k = 1$).

Proposition 21 *Assuming Conjecture* 1 *we have*

$$\delta(c_{p-1}(1) = j) = \begin{cases} \frac{A}{2} & \textit{if } j = -1; \\ 1 - A & \textit{if } j = 0; \\ \frac{A}{2} & \textit{if } j = 1. \end{cases}$$

Numerically we find up to the first 10^6 primes 'densities' of 0.18732, 0.625881 and 0.186799 respectively (for $-1, 0, 1$). This should be compared to the conjectural values $0.1869\cdots$, $0.6260\cdots$ and $0.1869\cdots$, respectively.

We demonstrate part 1 of Theorem 5 in Table 6 for $k = 15$, where we took x such that $\pi(x) = 10^6$ and rounded the result in the column 'numerical' to the sixth decimal.

Table 6: Density of values of $|S_{15}(p)|$ mod p

v	theoretical	numerical	approximate
0	$1 - 561A/475$	0.558339	0.558178
1	$9A/19$	0.177137	0.177157
2	$6A/19$	0.118091	0.118138
3	$2A/19$	0.039364	0.039353
4	$12A/95$	0.047237	0.047316
5	$12A/475$	0.009447	0.009457
8	$8A/95$	0.031491	0.031508
10	$8A/475$	0.006298	0.006317
12	$8A/285$	0.010497	0.010486
15	$8A/1425$	0.002100	0.002090

Let r be a prime and let $k = r^e$ for some $e \ge 0$. By Lemma 4 and (5) we deduce

$$c_{p-1}(k) = \begin{cases} \mu(p-1) & \text{if } \nu_r(p-1) = 0; \\ r^f(1-\frac{1}{r})\mu(\frac{p-1}{r^f}) & \text{if } \nu_r(p-1) = f,\ 1 \le f \le e; \\ r^e\mu(\frac{p-1}{r^e}) & \text{if } \nu_r(p-1) > e. \end{cases} \tag{34}$$

As regards to the average size of $c_{p-1}(k)$ we have the following result.

Theorem 6 *Let k be any natural number, and H any positive real number. Then*

$$\sum_{p\le x} |c_{p-1}(k)| = A\prod_{q|k}\left(1+\frac{\nu_q(k)(q-1)^2}{q^2-q-1}\right)\mathrm{Li}(x)+O\left(\frac{x}{\log^H x}\right),$$

where the O-constant depends at most on k and H.

Corollary 3 *The average of $|c_{p-1}(k)|/A$ over the primes p is a multiplicative function in k.*

Thus the average of $|c_{p-1}(k)|$ is a semi-multiplicative function in k. The analog of this for the natural number case also holds true: $M_n(|c_n(m)|)$ is a semi-multiplicative function in m (we leave this as an exercise to the reader).

Proof of Theorem 6. Let us denote by $\delta(q,e)$ the density of primes p such that $\nu_q(p-1)=e$. By Lemma 3 one infers that

$$\delta(q,e)=\begin{cases}1-\frac{1}{q-1} & \text{if } e=0;\\ q^{-e} & \text{if } e\ge 1.\end{cases}$$

By property 6 of Proposition 3, Proposition 6, the remark following Proposition 6 and (34), we infer that the theorem holds with a constant given by

$$\delta(\nu_{S_k}(p-1)\ne 0)\prod_{q|k}\left(\delta(q,0)+\sum_{1\le f\le \nu_q(k)}\varphi(q^f)\delta(q,f)+q^{\nu_q(k)}\delta(q,\nu_q(k)+1)\right).$$

On evaluating this constant, the result follows. □

Theorem 6 is demonstrated in Table 7 (with again x such that $\pi(x)=10^6$).

Table 7: Average of $|c_{p-1}(k)|$

k	theoretical	numerical	approximate
8	$4A$	1.494779	1.495823
21	$693A/205$	1.264572	1.264153
24	$36A/5$	2.689772	2.692482
27	$17A/5$	1.272214	1.271450
30	$126A/19$	2.479323	2.479918
36	$39A/5$	2.917172	2.916855

Our most general result in this section concerns the moments of $c_{p-1}(k)$.

Theorem 7 *Let k be a natural number.*
1). *Let $z\ne 1$ be a positive real number and H be a positive real number. Then*

$$\sum_{p\le x}|c_{p-1}(k)|^z=$$

$$A\prod_{q|k}\left(1+\frac{(q^{\nu_q(k)(z-1)}-1)(q-1)[(q-1)^z+q^{z-1}-1]}{(q^2-q-1)(q^{z-1}-1)}\right)\mathrm{Li}(x)+O\left(\frac{x}{\log^H x}\right),$$

where the O-constant depends at most on k and H.
2). *If we assume Conjecture 1 and j is an odd natural number, then we have $\sum_{p\le x}c_{p-1}(k)^j=o(\pi(x))$, where the o-constant depends at most on k and j.*

Proof. Part 1. Proceeding as in the proof of Theorem 6 we infer that the result holds with constant

$$\delta(\nu_{S_k}(p-1)\neq 0)\prod_{q|k}\left(\delta(q,0)+\sum_{1\le f\le \nu_q(k)}\varphi(q^f)^{2j}\delta(q,f)+q^{2j\nu_q(k)}\delta(q,\nu_q(k)+1)\right),$$

which is easily seen to equal the claimed constant.
Part 2. Similar to the proof of Theorem 5. □

Remark. If in part 1 we take the limit for z tending to one, then we obtain Theorem 6.

9.3 Value distribution of $s_k(p)$ mod p

Recall that $s_k(p)\equiv(-1)^k a_{p-1}(k)(\text{mod }p)$. It follows that $|\{s_k(p)\text{ mod }p\}|\le|\{a_n(k):n\ge 1\}|$. A presumably difficult question is to investigate under which conditions equality holds here. A related question is what the set $\{a_{p-1}(k):k\ge 1,\ p\text{ prime }\}$ looks like (cf. Suzuki's result in Section 4.1).

First we consider the value distribution of $s_k(p)$ for some small values of k. When $k=1$ we have $s_1(p)=S_1(p)$ and hence for this case we refer to the previous section. Note that $s_2(p)=\sum_{1\le i<j\le\varphi(p-1)}g_ig_j$. In the remainder of this section we assume that $p>2$ (some of the assertions will also be valid for $p=2$). Using Lemma 4 we infer that

$$s_2(p)=\frac{1}{2}(S_1(p)^2-S_2(p))\equiv\frac{1}{2}\left((\mu(p-1))^2-\frac{\varphi(p-1)}{\varphi(\frac{p-1}{2})}\mu(\frac{p-1}{2})\right)(\text{mod }p);$$

where the latter identity is Theorem 3 of [5]. By (5) we have

$$\frac{\varphi(p-1)}{\varphi(\frac{p-1}{2})}=\begin{cases}2 & \text{if } p\equiv 1(\text{mod }4);\\ 1 & \text{if } p\equiv 3(\text{mod }4).\end{cases}$$

On using this we infer that

$$s_2(p)\equiv\begin{cases}-\mu(\frac{p-1}{2}) & \text{if } p\equiv 1(\text{mod }4);\\ \mu(p-1)(\mu(p-1)+1)/2 & \text{if } p\equiv 3(\text{mod }4).\end{cases}\tag{35}$$

(For notational convenience we will write $s_k(p)\equiv a$ for $s_k(p)\equiv a(\text{mod }p)$.) Since $s_2(2)=0$ it follows that the reduction of $s_2(p)$ mod p is in $\{-1,0,1\}$. The following assertion conjecturally resolves the question of the Dence brothers stated in the introduction.

Proposition 22 *We have, assuming the validity of Conjecture* 1,

$$\delta(s_2(p) \equiv j) = \begin{cases} \frac{A}{4} & \text{if } j = -1; \\ 1 - A & \text{if } j = 0; \\ \frac{3}{4}A & \text{if } j = 1. \end{cases}$$

Proof. Equation (35) suggests to consider the primes p with $p \equiv 1 (\text{mod } 4)$ and $p \equiv 3(\text{mod } 4)$ separately. Note that for $x \geq 2$,

$$\sum_{\substack{p \leq x,\ p \equiv 3(\text{mod } 4) \\ \mu(p-1)=1}} 1 = -1 + \sum_{\substack{p \leq x \\ \mu(p-1)=1}} 1.$$

The latter quantity we (conditionally) evaluated in Proposition 21. It follows that the density of primes p such that $p \equiv 3(\text{mod } 4)$ and $s_2(p) \equiv 1$ equals $A/2$ and that the density of primes p such that $p \equiv 3(\text{mod } 4)$ and $s_2(p) \equiv 0$ equals $1/2 - A/2$.

The case $p \equiv 1(\text{mod } 4)$ can be dealt with Proposition 6 (with $t = 1$, $q_1 = 2$ and $e_1 = 2$) and Conjecture 1. The results are summed up in Table 8. □

Table 8: Value distribution of $s_2(p)$ mod p

δ	-1	0	1	$+$
$\nu_2(p-1) \leq 1$	0	$1/2 - A/2$	$A/2$	$1/2$
numerical	0.000000	0.313022	0.186978	0.500000
$\pi(x) = 10^6$	0.000000	0.313403	0.186798	0.500201
$\nu_2(p-1) = 2$	$A/4$	$1/2 - A/2$	$A/4$	$1/2$
numerical	0.093489	0.313022	0.093489	0.500000
$\pi(x) = 10^6$	0.093939	0.312813	0.093047	0.499799
$+$	$A/4$	$1 - A$	$3A/4$	1
numerical	0.093489	0.626044	0.280467	1.000000
$\pi(x) = 10^6$	0.093939	0.626216	0.279845	1.000000

By Newton's formula $s_k(p)$ can be expressed as a polynomial in $S_r(p)$ with $1 \leq r \leq k$. To be precise, we have

$$(-1)^k s_k(p) = \sum \frac{(-1)^{k_1+k_2+\cdots}}{k_1! 1^{k_1} k_2! 2^{k_2} \cdots} S_1(p)^{k_1} S_2(p)^{k_2} \cdots, \tag{36}$$

where the sum extends over all solutions $(k_1, k_2, \cdots)$ of $k_1 + 2k_2 + \cdots = k$. Next we consider $s_3(p)$. Taking $k = 3$ in (36) we compute

$$s_3(p) = \frac{S_1(p)^3 + 2S_3(p) - 3S_1(p)S_2(p)}{6}.$$

Invoking Lemma 4 a more explicit formula for $s_3(p)$ can then be derived. With $\beta = \nu_3(p-1)$ we find

$$s_3(p) = \begin{cases} \mu(p-1)(\mu(p-1)+1)/2 & \text{if } \beta = 0; \\ \mu(p-1)(\mu(p-1)-1)/2 & \text{if } \beta = 1; \\ \mu(\frac{p-1}{3}) & \text{if } \beta \geq 2. \end{cases}$$

In each of these cases Proposition 6 and Conjecture 1 yield the densities with which the values $-1, 0$ and 1 are assumed. The results are summed up in Table 9.

Table 9: Value distribution of $s_3(p)$

δ	-1	0	1	$+$
$\beta = 0$	0	$1/2 - 3A/10$	$3A/10$	$1/2$
numerical	0.000000	0.387813	0.112187	0.500000
$\pi(x) = 10^6$	0.000000	0.388136	0.112035	0.500171
$\beta = 1$	0	$1/3 - A/5$	$A/5$	$1/3$
numerical	0.000000	0.258542	0.074791	0.333333
$\pi(x) = 10^6$	0.000000	0.258378	0.074883	0.333261
$\beta \geq 2$	$A/15$	$1/6 - 2A/15$	$A/15$	$1/6$
numerical	0.024930	0.116806	0.024930	0.166666
$\pi(x) = 10^6$	0.025031	0.116729	0.024809	0.166569
$+$	$A/15$	$1 - 19A/30$	$17A/30$	1
numerical	0.024930	0.763161	0.211908	1.000000
$\pi(x) = 10^6$	0.025030	0.763243	0.211727	1.000000

Adding up the various contributions we infer the following result.

Proposition 23 *We have, assuming the validity of Conjecture* 1,

$$\delta(s_3(p) \equiv j) = \begin{cases} \frac{A}{15} & \textit{if } j = -1; \\ 1 - \frac{19}{30}A & \textit{if } j = 0; \\ \frac{17}{30}A & \textit{if } j = 1. \end{cases}$$

Similarly, for $s_4(p)$ we find, with $\alpha = \nu_2(p-1)$ and $\beta = \nu_3(p-1)$,

$$s_4(p) \equiv \begin{cases} \mu(p-1)(\mu(p-1)+1)/2 & \text{if } \alpha = 1 \text{ and } \beta = 0; \\ \mu(p-1)(1-\mu(p-1))/2 & \text{if } \alpha = 1 \text{ and } \beta \geq 1; \\ \mu(\frac{p-1}{2})(\mu(\frac{p-1}{2})+1)/2 & \text{if } \alpha = 2; \\ -\mu(\frac{p-1}{4}) & \text{if } \alpha \geq 3. \end{cases}$$

Proceeding as before we infer the following result.

Proposition 24 *We have, assuming the validity of Conjecture* 1,

$$\delta(s_4(p) \equiv j) = \begin{cases} \frac{13}{40}A & \textit{if } j = -1; \\ 1 - A & \textit{if } j = 0; \\ \frac{27}{40}A & \textit{if } j = 1. \end{cases}$$

The above approach of dealing with the value distribution of $s_k(p) (\mathrm{mod}\ p)$ for small p clearly becomes more laborious as k increases. The following result is more systematic; it is analogous to Theorem 4.

Theorem 8 *Let $k \geq 1$ be fixed. Let $M_k = k\prod_{p \leq k} p$ and let $r > k$ be any prime. Then*

$$\sum_{p \leq x} a_{p-1}(k) = \frac{Ax}{2\log x} \prod_{2 < q \leq k} \frac{q(q-2)}{q^2-q-1} \sum_{\substack{d | M_k \\ 2 | d}} \frac{a_d(k) + a_{dr}(k)}{d} \prod_{\substack{q | d \\ q > 2}} \frac{q-1}{q-2} + o(\frac{x}{\log x}).$$

If $v \neq 0$, then

$$\sum_{\substack{p\le x\\ a_{p-1}(k)=v}} 1 = B_2(k)\frac{x}{\log x} + o(\frac{x}{\log x}),$$

where

$$B_2(k) = \frac{A}{2}\prod_{2<q\le k}\frac{q(q-2)}{q^2-q-1}\left(\sum_{\substack{d|M_k,\ 2|d\\ a_d(k)=v}}\frac{1}{d}\prod_{\substack{q|d\\ q>2}}\frac{q-1}{q-2} + \sum_{\substack{d|M_k,\ 2|d\\ a_{dr}(k)=v}}\frac{1}{d}\prod_{\substack{q|d\\ q>2}}\frac{q-1}{q-2}\right).$$

In both cases the o-constant depends at most on k.

Proof. Put $N_k = \mathrm{lcm}(1,2,\cdots,k)\prod_{p\le k} p$. Let $S(k) = \{q_1,\ldots,q_{\pi(k)}\}$ be the set of primes not exceeding k and $r_2 = N_k\prod_{p\le k} p$. Furthermore, let $d|N_k$. We first consider $\sum_{p\le x,\ (p-1,r_2)=d}\mu((p-1)/d)$. Let $q\in S(k)$ and suppose that p satisfies $(p-1,r_2)=d$. Since $\nu_q(r_2)>\nu_q(d)$, it follows that $\nu_q(p-1)=\nu_q(d)$. Note that $\mu((p-1)/d)=\mu_{S(k)}(p-1)$. Conjecture 1 implies now that

$$\sum_{p\le x,\ (p-1,r_2)=d}\mu(\frac{p-1}{d}) = \sum_{\substack{p\le x\\ \nu_{S(k)}(p-1)=(\nu_{q_1}(d),\ldots,\nu_{q_{\pi(k)}}(d))}}\mu_{S(k)}(p-1) = o(\pi(x)), \quad (37)$$

where the o-constant depends at most on k.

Next we consider

$$\sum_{\substack{p\le x\\ (p-1,r_2)=d}}\mu(\frac{p-1}{d})^2 = \sum_{\substack{p\le x\\ \nu_{S(k)}(p-1)=(\nu_{q_1}(d),\ldots,\nu_{q_{\pi(k)}}(d))}}\mu_{S(k)}(p-1)^2. \quad (38)$$

By Proposition 6 we then find that

$$\sum_{\substack{p\le x\\ (p-1,r)=d}}\mu(\frac{p-1}{d})^2 = \frac{1}{d}\prod_{\substack{q\le k\\ q\nmid d}}\left(1-\frac{1}{q-1}\right)\prod_{q>k}\left(1-\frac{1}{q(q-1)}\right)\frac{x}{\log x} + o(\frac{x}{\log x}).$$

Using Proposition 7 we have

$$\sum_{p\le x} a_{p-1}(k) = \sum_{d|N_k}\sum_{\substack{p\le x\\ (p-1,r_2)=d}}\left(A_1(d)\mu(\frac{p-1}{d})^2 + B_1(d)\mu(\frac{p-1}{d})\right).$$

From the latter formula in combination with (37) and (38), we infer that

$$\sum_{p\le x} a_{p-1}(k) = \frac{Ax}{\log x}\prod_{2<q\le k}\frac{q(q-2)}{q^2-q-1}\sum_{\substack{d|N_k\\ 2|d}}\frac{A_1(d)}{d}\prod_{\substack{q|d\\ q>2}}\frac{q-1}{q-2} + o(\frac{x}{\log x}).$$

On noting that $A_1(d) = (a_d(k)+a_{dr}(k))/2$ and using (10), the proof of the first assertion is completed. The proof of the second assertion is a variation of the argument above and left to the reader. $\square$

The latter theorem is demonstrated in Table 10. Using that the sum of the densities equals one, one can easily find the density of primes for which $a_{p-1}(k) = 0$ from the table, so we omitted this information. Notice that the averages given for $1 \le k \le 4$ are the same as can be inferred from the earlier presented results on the value distribution of $S_1(p), s_2(p), s_3(p)$ and $s_4(p)$ respectively. The averages given also seem to be consistent with numerical simulations.

Table 10: Conjectural value distribution of $\delta(a_{p-1}(k) = v)/A$

j	-2	-1	1	2	Average
$k = 1$	0	1/2	1/2	0	0
$k = 2$	0	1/4	3/4	0	1/2
$k = 3$	0	17/30	1/15	0	$-1/2$
$k = 4$	0	13/40	27/40	0	7/20
$k = 5$	0	69/90	6/95	0	$-3/10$
$k = 6$	0	443/1140	47/95	0	121/1140
$k = 7$	0	13989/54530	358/1435	24/3895	1/190
$k = 8$	0	16703/62320	35873/62320	0	1917/6232
$k = 9$	0	31477/70110	2129/35055	0	$-9073/23370$
$k = 10$	0	267/820	505/1558	0	$-23/15580$

It is not difficult to see that $\delta(a_{p-1}(k) = v) > 0$ if and only if $a_{p-1}(k) = v$ for some prime p. We infer from the table that there is no prime p such that $a_{p-1}(7) = -2$, whereas by Table 4 it follows that $\delta(a_n(7) = -2) = 0.001055\cdots$. Indeed, the example following Proposition 11 shows that there are no even integers n for which $a_n(7) = -2$.

10 Open problems

During research usually more questions are being raised than being resolved and the present work is no exception. We conclude by formulating some questions. Gallot's data show that 2, 5 and 9 hold true for $k \le 100$.

1) Is Conjecture 1 true ?
2) Is Möller's conjecture that $0 \le e_k \le 1/2$ true ?
3) What is the behaviour of e_k for large k ?
4) Compute $M(e_k)$.
5) Is $e_k k \prod_{p \le k}(p+1) \in \mathbb{Z}$?
6) Determine $\{a_{p-1}(k) : p \text{ prime}, k \in \mathbb{N}\}$.
7) Does $a_n(k) = -v$, $v \le 0$, imply there is an integer m with $a_m(k) = v$?
8) Is $|\mathcal{B}(k) \backslash \mathcal{B}_0(k)| \ge 1$ infinitely often ?
9) Is $\delta(a_n(k) = 1) \ge \delta(a_n(k) = -1)$?
10) What is the behaviour of $\delta(a_n(k) \notin \{-1, 0, 1\})$ for large k ?

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Table 11: Value distribution of $a_n(k)$ for $1 \le k \le 61$

(The number between brackets is $k\prod_{p\le k}(p+1)e_k$.)

$e_1 = 0(0.000000)[0]$
$V_1[-1] = 1/2(0.500000)$
$V_1[+1] = 1/2(0.500000)$

$e_2 = 1/2(0.500000)[3]$
$V_2[-1] = 1/12(0.083333)$
$V_2[+1] = 7/12(0.583333)$

$e_3 = 1/6(0.166667)[6]$
$V_3[-1] = 5/24(0.208333)$
$V_3[+1] = 3/8(0.375000)$

$e_4 = 1/3(0.333333)[16]$
$V_4[-1] = 1/6(0.166667)$
$V_4[+1] = 1/2(0.500000)$

$e_5 = 1/8(0.125000)[45]$
$V_5[-1] = 13/80(0.162500)$
$V_5[+1] = 23/80(0.287500)$

$e_6 = 7/24(0.291667)[126]$
$V_6[-1] = 25/144(0.173611)$
$V_6[+1] = 67/144(0.465278)$

$e_7 = 1/18(0.055556)[224]$
$V_7[-2] = 1/576(0.001736)$
$V_7[-1] = 577/2688(0.214658)$
$V_7[+1] = 731/2688(0.271949)$
$V_7[+2] = 1/1152(0.000868)$

$e_8 = 7/24(0.291667)[1344]$
$V_8[-1] = 1/8(0.125000)$
$V_8[+1] = 5/12(0.416667)$

$e_9 = 19/144(0.131944)[684]$
$V_9[-1] = 65/384(0.169271)$
$V_9[+1] = 347/1152(0.301215)$

$e_{10} = 31/160(0.193750)[1116]$
$V_{10}[-1] = 161/960(0.167708)$
$V_{10}[+1] = 347/960(0.361458)$

$e_{11} = 1/16(0.062500)[4752]$
$V_{11}[-2] = 1/2304(0.000434)$
$V_{11}[-1] = 8299/50688(0.163727)$
$V_{11}[+1] = 11489/50688(0.226661)$
$V_{11}[+2] = 1/4608(0.000217)$

$e_{12} = 55/192(0.286458)[23760]$
$V_{12}[-1] = 349/2304(0.151476)$
$V_{12}[+1] = 1009/2304(0.437934)$

$e_{13} = 13/288(0.045139)[56784]$
$V_{13}[-2] = 43/48384(0.000889)$
$V_{13}[-1] = 219269/1257984(0.174302)$
$V_{13}[+1] = 277171/1257984(0.220330)$
$V_{13}[+2] = 43/96768(0.000444)$

$e_{14} = 61/288(0.211806)[286944]$
$V_{14}[-2] = 13/21504(0.000605)$
$V_{14}[-1] = 2395/21504(0.111375)$
$V_{14}[+1] = 2319/7168(0.323521)$
$V_{14}[+2] = 1/2304(0.000434)$

$e_{15} = 2287/20160(0.113442)[164664]$
$V_{15}[-2] = 13/32256(0.000403)$
$V_{15}[-1] = 1345/7168(0.187640)$
$V_{15}[+1] = 97247/322560(0.301485)$
$V_{15}[+2] = 13/64512(0.000202)$

$e_{16} = 733/4032(0.181796)[281472]$
$V_{16}[-2] = 5/21504(0.000233)$
$V_{16}[-1] = 12149/64512(0.188322)$
$V_{16}[+1] = 1127/3072(0.366862)$
$V_{16}[+2] = 5/2688(0.001860)$

$e_{17} = 667/8064(0.082713)[2449224]$
$V_{17}[-3] = 5/580608(0.000009)$
$V_{17}[-2] = 281/193536(0.001452)$
$V_{17}[-1] = 2353487/19740672(0.119220)$
$V_{17}[+1] = 1981753/9870336(0.200779)$
$V_{17}[+2] = 197/96768(0.002036)$
$V_{17}[+3] = 5/1161216(0.000004)$

$e_{18} = 79/336(0.235119)[7371648]$
$V_{18}[-2] = 961/1161216(0.000828)$
$V_{18}[-1] = 5575/43008(0.129627)$
$V_{18}[+1] = 212369/580608(0.365770)$
$V_{18}[+2] = 341/1161216(0.000294)$
$V_{18}[+3] = 17/1161216(0.000015)$

$e_{19} = 55/1344(0.040923)[27086400]$
$V_{19}[-3] = 19/2322432(0.000008)$
$V_{19}[-2] = 67813/69672960(0.000973)$
$V_{19}[-1] = 42731243/264757248(0.161398)$
$V_{19}[+1] = 67086449/330946560(0.202711)$
$V_{19}[+2] = 54641/69672960(0.000784)$
$V_{19}[+3] = 19/4644864(0.000004)$

$e_{20} = 221/960(0.230208)[160392960]$
$V_{20}[-3] = 1/221184(0.000005)$
$V_{20}[-2] = 307/165888(0.001851)$
$V_{20}[-1] = 1417037/11612160(0.122030)$
$V_{20}[+1] = 8240789/23224320(0.354834)$
$V_{20}[+2] = 361/645120(0.000560)$

$e_{21} = 8207/120960(0.067849)[49635936]$
$V_{21}[-3] = 1/331776(0.000003)$
$V_{21}[-2] = 138259/69672960(0.001984)$
$V_{21}[-1] = 14659501/69672960(0.210404)$
$V_{21}[+1] = 69503/248832(0.279317)$
$V_{21}[+2] = 101363/69672960(0.001455)$
$V_{21}[+3] = 1/663552(0.000002)$

$e_{22} = 8467/95040(0.089089)[68277888]$
$V_{22}[-3] = 1/221184(0.000005)$
$V_{22}[-2] = 7/18432(0.000380)$
$V_{22}[-1] = 8101001/42577920(0.190263)$
$V_{22}[+1] = 70728809/255467520(0.276860)$
$V_{22}[+2] = 3751/2322432(0.001615)$
$V_{22}[+3] = 19/1658880(0.000011)$

$e_{23} = 629/11520(0.054601)[1049956992]$
$V_{23}[-4] = 1/7962624(0.000000)$
$V_{23}[-3] = 107/7962624(0.000013)$
$V_{23}[-2] = 578371/418037760(0.001384)$
$V_{23}[-1] = 5016105893/38459473920(0.130426)$
$V_{23}[+1] = 1789013287/9614868480(0.186067)$
$V_{23}[+2] = 730129/836075520(0.000873)$
$V_{23}[+3] = 107/15925248(0.000007)$
$V_{23}[+4] = 1/15925248(0.000000)$

$e_{24} = 7327/24192(0.302869)[6077306880$
$V_{24}[-3] = 853/69672960(0.000012)$
$V_{24}[-2] = 247859/278691840(0.000889)$
$V_{24}[-1] = 898117/9289728(0.096679)$
$V_{24}[+1] = 55697567/139345920(0.399707)$
$V_{24}[+2] = 8227/10321920(0.000797)$
$V_{24}[+3] = 713/34836480(0.000020)$

$e_{25} = 1087/18144(0.059910)[1252224000]$
$V_{25}[-3] = 25/7962624(0.000003)$
$V_{25}[-2] = 81607/238878720(0.000342)$
$V_{25}[-1] = 1375923029/8360755200(0.164569)$
$V_{25}[+1] = 469535011/2090188800(0.224638)$
$V_{25}[+2] = 88489/334430208(0.000265)$
$V_{25}[+3] = 25/15925248(0.000002)$

$e_{26} = 234433/1572480(0.149085)[3240801792]$
$V_{26}[-3] = 25/5308416(0.000005)$
$V_{26}[-2] = 11149/15482880(0.000720)$
$V_{26}[-1] = 837724879/7245987840(0.115612)$
$V_{26}[+1] = 127807265/483065856(0.264575)$
$V_{26}[+2] = 54667/69672960(0.000785)$
$V_{26}[+3] = 1289/557383680(0.000002)$

$e_{27} = 33491/362880(0.092292)[2083408128]$
$V_{27}[-3] = 811/185794560(0.000004)$
$V_{27}[-2] = 311893/836075520(0.000373)$
$V_{27}[-1] = 244503443/1672151040(0.146221)$
$V_{27}[+1] = 399281329/1672151040(0.238783)$
$V_{27}[+2] = 201713/836075520(0.000241)$
$V_{27}[+3] = 139/61931520(0.000002)$

$e_{28} = 84047/483840(0.173708)[4066530048]$
$V_{28}[-3] = 1073/69672960(0.000015)$
$V_{28}[-2] = 202387/278691840(0.000726)$
$V_{28}[-1] = 5770579/39813120(0.144942)$
$V_{28}[+1] = 88832659/278691840(0.318749)$
$V_{28}[+2] = 96629/139345920(0.000693)$
$V_{28}[+3] = 1/241920(0.000004)$
$V_{28}[+4] = 1/7741440(0.000000)$

$e_{29} = 5021/103680(0.048428)[35225729280]$
$V_{29}[-4] = 1837/25082265600(0.000000)$
$V_{29}[-3] = 215897/16721510400(0.000013)$
$V_{29}[-2] = 56528869/50164531200(0.001127)$
$V_{29}[-1] = 5288720741/41564897280(0.127240)$

$V_{29}[+1] = 256004923169/1454771404800(0.175976)$
$V_{29}[+2] = 48779039/50164531200(0.000972)$
$V_{29}[+3] = 24511/1857945600(0.000013)$
$V_{29}[+4] = 3883/50164531200(0.000000)$

$e_{30} = 45893/241920(0.189703)[142745587200]$
$V_{30}[-3] = 55183/8360755200(0.000007)$
$V_{30}[-2] = 2194267/2090188800(0.001050)$
$V_{30}[-1] = 541742161/3344302080(0.161990)$
$V_{30}[+1] = 5867204267/16721510400(0.350878)$
$V_{30}[+2] = 34039/23887872(0.001425)$
$V_{30}[+3] = 450059/16721510400(0.000027)$
$V_{30}[+4] = 8107/8360755200(0.000001)$
$V_{30}[+5] = 29/5573836800(0.000000)$

$e_{31} = 155/5376(0.028832)[717382656000]$
$V_{31}[-4] = 341/7644119040(0.000000)$
$V_{31}[-3] = 25652623/802632499200(0.000032)$
$V_{31}[-2] = 1728495157/1605264998400(0.001077)$
$V_{31}[-1] = 4263256278479/24881607475200(0.171342)$
$V_{31}[+1] = 4956093793291/24881607475200(0.199187)$
$V_{31}[+2] = 2463624983/1605264998400(0.001535)$
$V_{31}[+3] = 44559467/802632499200(0.000056)$
$V_{31}[+4] = 341/15288238080(0.000000)$

$e_{32} = 4381/17920(0.244475)[6279166033920]$
$V_{32}[-4] = 143/5096079360(0.000000)$
$V_{32}[-3] = 143131/19818086400(0.000007)$
$V_{32}[-2] = 32276503/38220595200(0.000844)$
$V_{32}[-1] = 45508919251/535088332800(0.085049)$
$V_{32}[+1] = 24918610681/76441190400(0.325984)$
$V_{32}[+2] = 667940893/267544166400(0.002497)$
$V_{32}[+3] = 8249729/107017666560(0.000077)$
$V_{32}[+4] = 1207481/178362777600(0.000007)$

$e_{33} = 294509/3193344(0.092226)[2442775449600]$
$V_{33}[-4] = 20065/10701766656(0.000002)$
$V_{33}[-3] = 80453581/802632499200(0.000100)$
$V_{33}[-2] = 412044869/401316249600(0.001027)$
$V_{33}[-1] = 2239952947681/17657914982400(0.126853)$
$V_{33}[+1] = 240607474937/1103619686400(0.218017)$
$V_{33}[+2] = 1136159459/802632499200(0.001416)$
$V_{33}[+3] = 309052627/1605264998400(0.000193)$
$V_{33}[+4] = 398297/107017666560(0.000004)$

$e_{34} = 268801/3525120(0.076253)[2080906813440]$
$V_{34}[-4] = 143/5096079360(0.000000)$
$V_{34}[-3] = 20977/5096079360(0.000004)$
$V_{34}[-2] = 1313801/1189085184(0.001105)$
$V_{34}[-1] = 1500096902327/9096501657600(0.164909)$
$V_{34}[+1] = 135360848387/568531353600(0.238089)$
$V_{34}[+2] = 80945987/33443020800(0.002420)$
$V_{34}[+3] = 3130531/21403533312(0.000146)$
$V_{34}[+4] = 1071667/267544166400(0.000004)$
$V_{34}[+5] = 719/19818086400(0.000000)$

$e_{35} = 69809/870912(0.080156)[2251759104000]$
$V_{35}[-5] = 323/2090188800(0.000000)$
$V_{35}[-4] = 242639/25082265600(0.000010)$
$V_{35}[-3] = 280214099/1605264998400(0.000175)$
$V_{35}[-2] = 428425901/229323571200(0.001868)$
$V_{35}[-1] = 15515446957/107017666560(0.144980)$
$V_{35}[+1] = 120969894043/535088332800(0.226075)$
$V_{35}[+2] = 2211435001/1605264998400(0.001378)$
$V_{35}[+3] = 292252153/1605264998400(0.000182)$
$V_{35}[+4] = 4691237/321052999680(0.000015)$
$V_{35}[+5] = 323/1045094400(0.000000)$

$e_{36} = 41333/207360(0.199330)[5759584911360]$
$V_{36}[-3] = 5072809/76441190400(0.000066)$
$V_{36}[-2] = 31579379/26754416640(0.001180)$
$V_{36}[-1] = 92196104039/535088332800(0.172301)$
$V_{36}[+1] = 12419782037/33443020800(0.371371)$
$V_{36}[+2] = 124255577/107017666560(0.001161)$
$V_{36}[+3] = 78450899/535088332800(0.000147)$
$V_{36}[+4] = 22711/1651507200(0.000014)$
$V_{36}[+5] = 56381/178362777600(0.000000)$
$V_{36}[+6] = 14633/535088332800(0.000000)$

$e_{37} = 19073/544320(0.035040)[39542741729280]$
$V_{37}[-5] = 30631/3210529996800(0.000000)$
$V_{37}[-4] = 56904067/61000069939200(0.000001)$
$V_{37}[-3] = 60188399/1742859141120(0.000035)$
$V_{37}[-2] = 76311280099/61000069939200(0.001251)$
$V_{37}[-1] = 323928333998587/2257002587750400(0.143521)$
$V_{37}[+1] = 80619151138297/451400517550080(0.178598)$
$V_{37}[+2] = 73686178829/61000069939200(0.001208)$
$V_{37}[+3] = 763875323/15250017484800(0.000050)$
$V_{37}[+4] = 51705001/30500034969600(0.000002)$
$V_{37}[+5] = 30631/1605264998400(0.000000)$

$e_{38} = 246539/1451520(0.169849)[196855040655360]$
$V_{38}[-4] = 7955363/2541669580800(0.000003)$
$V_{38}[-3] = 18756869/150617456640(0.000125)$
$V_{38}[-2] = 1660704037/1694446387200(0.000980)$
$V_{38}[-1] = 137023551787/1694446387200(0.080866)$
$V_{38}[+1] = 1265500142927/5083339161600(0.248951)$
$V_{38}[+2] = 4927088377/2541669580800(0.001939)$
$V_{38}[+3] = 96190153/1270834790400(0.000076)$
$V_{38}[+4] = 17066753/10166678323200(0.000002)$
$V_{38}[+5] = 629/356725555200(0.000000)$

$e_{39} = 961879/12804480(0.075121)[89355942789120]$
$V_{39}[-6] = 1/179159040(0.000000)$
$V_{39}[-5] = 179/627056640(0.000000)$
$V_{39}[-4] = 77322241/12200013987840(0.000006)$
$V_{39}[-3] = 6774345563/61000069939200(0.000111)$
$V_{39}[-2] = 2729296757/2033335664640(0.001342)$
$V_{39}[-1] = 121853678359577/793000909209600(0.153661)$
$V_{39}[+1] = 181373355248401/793000909209600(0.228718)$
$V_{39}[+2] = 13458490453/10166678323200(0.001324)$
$V_{39}[+3] = 1683824087/12200013987840(0.000138)$
$V_{39}[+4] = 7100759/642105999360(0.000011)$
$V_{39}[+5] = 179/313528320(0.000001)$
$V_{39}[+6] = 1/89579520(0.000000)$

$e_{40} = 3360923/18385920(0.182799)[223014717849600]$
$V_{40}[-5] = 198679/26754416640(0.000007)$
$V_{40}[-4] = 2961557/4066671329280(0.000001)$
$V_{40}[-3] = 5036034173/20333356646400(0.000248)$
$V_{40}[-2] = 30381984071/20333356646400(0.001494)$
$V_{40}[-1] = 2957102734199/20333356646400(0.145431)$
$V_{40}[+1] = 3360988664941/10166678323200(0.330589)$
$V_{40}[+2] = 6478070507/10166678323200(0.000637)$
$V_{40}[+3] = 292144933/6777785548800(0.000043)$
$V_{40}[+4] = 9938893/5083339161600(0.000002)$
$V_{40}[+5] = 4597979/20333356646400(0.000000)$
$V_{40}[+6] = 13/30576476160(0.000000)$

$e_{41} = 7313311/165473280(0.044196)[2321237890621440]$
$V_{41}[-6] = 61468511/512400587489280(0.000000)$
$V_{41}[-5] = 905963641/183000209817600(0.000005)$
$V_{41}[-4] = 739141561/128100146872320(0.000006)$
$V_{41}[-3] = 95451411817/427000489574400(0.000224)$
$V_{41}[-2] = 184063646579/73200083927040(0.002515)$
$V_{41}[-1] = 3161605893907487/26260530108825600(0.120394)$
$V_{41}[+1] = 8685630577953149/52521060217651200(0.165374)$
$V_{41}[+2] = 2865735759811/1281001468723200(0.002237)$

$V_{41}[+3] = 65007805063/427000489574400(0.000152)$
$V_{41}[+4] = 12935415793/2562002937446400(0.000005)$
$V_{41}[+5] = 46277477/18300020981760(0.000003)$
$V_{41}[+6] = 77476183/1281001468723200(0.000000)$

$e_{42} = 6123037/27578880(0.222019)[11945083135426560]$
$V_{42}[-6] = 396751/30500034969600(0.000000)$
$V_{42}[-5] = 14830471/20333356646400(0.000001)$
$V_{42}[-4] = 1896873157/142333496524800(0.000013)$
$V_{42}[-3] = 3201729511/13343765299200(0.000240)$
$V_{42}[-2] = 33789705539/18977799536640(0.001780)$
$V_{42}[-1] = 93023983380571/854000979148800(0.108927)$
$V_{42}[+1] = 283373054708011/854000979148800(0.331818)$
$V_{42}[+2] = 705994382021/427000489574400(0.001653)$
$V_{42}[+3] = 6305790613/122000139878400(0.000052)$
$V_{42}[+4] = 338310517/427000489574400(0.000001)$
$V_{42}[+5] = 159624961/854000979148800(0.000000)$
$V_{42}[+6] = 7585/1265186635776(0.000000)$

$e_{43} = 1241881/47278080(0.026268)[63663558680494080]$
$V_{43}[-7] = 899/20600900812800(0.000000)$
$V_{43}[-6] = 239374189/56364064623820800(0.000000)$
$V_{43}[-5] = 8097232133/56364064623820800(0.000000)$
$V_{43}[-4] = 35174564923/12525347694182400(0.000003)$
$V_{43}[-3] = 38747727775/501013907767296(0.000077)$
$V_{43}[-2] = 34634252980079/18788021541273600(0.001843)$
$V_{43}[-1] = 779722198548416059/4847309557648588800(0.160857)$
$V_{43}[+1] = 1906106119849907/10204862226628608(0.186784)$
$V_{43}[+2] = 8259199022633/4175115898060800(0.001978)$
$V_{43}[+3] = 536250514949/5368006154649600(0.000100)$
$V_{43}[+4] = 2687713733/751520861650944(0.000004)$
$V_{43}[+5] = 6777068293/56364064623820800(0.000000)$
$V_{43}[+6] = 281711701/112728129247641600(0.000000)$
$V_{43}[+7] = 899/41201801625600(0.000000)$

$e_{44} = 27117469/151683840(0.178776)[443368483462840320]$
$V_{44}[-7] = 19499/1569592442880(0.000000)$
$V_{44}[-6] = 483086393/521889487257600(0.000001)$
$V_{44}[-5] = 11256275861/596445128294400(0.000019)$
$V_{44}[-4] = 664972869923/7515208616509440(0.000088)$
$V_{44}[-3] = 1013158170881/12525347694182400(0.000081)$
$V_{44}[-2] = 35958299894189/12525347694182400(0.002871)$
$V_{44}[-1] = 209668413473641/2087557949030400(0.100437)$
$V_{44}[+1] = 279518297968507/988843239014400(0.282672)$
$V_{44}[+2] = 2998032785137/2087557949030400(0.001436)$
$V_{44}[+3] = 1257623773787/37576043082547200(0.000033)$
$V_{44}[+4] = 64724291587/37576043082547200(0.000002)$

$V_{44}[+5] = 50451869/1977686478028800(0.000000)$
$V_{44}[+6] = 65231/357867076976640(0.000000)$

$e_{45} = 3979615/45505152(0.087454)[221817245368320000]$
$V_{45}[-7] = 33263/18788021541273600(0.000000)$
$V_{45}[-6] = 566438617/10248011749785600(0.000000)$
$V_{45}[-5] = 269113485517/112728129247641600(0.000002)$
$V_{45}[-4] = 12278820607/4697005385318400(0.000003)$
$V_{45}[-3] = 151929636707/1503041723301888(0.000101)$
$V_{45}[-2] = 46254490783673/28182032311910400(0.001641)$
$V_{45}[-1] = 6148678815334457/37576043082547200(0.163633)$
$V_{45}[+1] = 295732896088667/1174251346329600(0.251848)$
$V_{45}[+2] = 149807819591587/112728129247641600(0.001329)$
$V_{45}[+3] = 2209747042441/37576043082547200(0.000059)$
$V_{45}[+4] = 340486901/191714505523200(0.000002)$
$V_{45}[+5] = 778779179/644160738557952(0.000001)$
$V_{45}[+6] = 314248783/11272812924764160(0.000000)$
$V_{45}[+7] = 33263/37576043082547200(0.000000)$

$e_{46} = 1634903647/27909826560(0.058578)[151878101868380160]$
$V_{46}[-7] = 593147/1404606873600(0.000000)$
$V_{46}[-6] = 8041/200658124800(0.000000)$
$V_{46}[-5] = 461160835213/37576043082547200(0.000012)$
$V_{46}[-4] = 87404403091/18788021541273600(0.000005)$
$V_{46}[-3] = 772557276649/18788021541273600(0.000041)$
$V_{46}[-2] = 14774599478917/6262673847091200(0.002359)$
$V_{46}[-1] = 30866961312674209/172849798179717120(0.178577)$
$V_{46}[+1] = 1461674766295439/6173207077847040(0.236777)$
$V_{46}[+2] = 22857774451559/9394010770636800(0.002433)$
$V_{46}[+3] = 533933483887/4175115898060800(0.000128)$
$V_{46}[+4] = 218641508917/18788021541273600(0.000012)$
$V_{46}[+5] = 2284159373/2087557949030400(0.000001)$
$V_{46}[+6] = 147831841/2684003077324800(0.000000)$
$V_{46}[+7] = 8611853/7515208616509440(0.000000)$
$V_{46}[+8] = 19/2809213747200(0.000000)$

$e_{47} = 64767749/1213470720(0.053374)[6786891421192028160]$
$V_{75}[-9] = 1147/715734153953280(0.000000)$
$V_{75}[-8] = 56233703/169092193871462400(0.000000)$
$V_{75}[-7] = 8196769949/676368775485849600(0.000000)$
$V_{75}[-6] = 231016391/27606888795340800(0.000000)$
$V_{75}[-5] = 115585824097/216438008155471872(0.000001)$
$V_{75}[-4] = 32560081364423/2705475101943398400(0.000012)$
$V_{75}[-3] = 31938251933447/193248221567385600(0.000165)$
$V_{75}[-2] = 10645199114119417/5410950203886796800(0.001967)$
$V_{75}[-1] = 27034488309751213037/254314659582679449600(0.106303)$
$V_{75}[+1] = 115130557033516291/722484828359884800(0.159354)$

$V_{75}[+2] = 5876022667634491/27054751019433984000(0.002172)$
$V_{75}[+3] = 31091772610243/216438008155471872(0.000144)$
$V_{75}[+4] = 3905889918467/541095020388679680(0.000007)$
$V_{75}[+5] = 191742884821/676368775485849600(0.000000)$
$V_{75}[+6] = 3669340891/772992886269542400(0.000000)$
$V_{75}[+7] = 32823296417/5410950203886796800(0.000000)$
$V_{75}[+8] = 56233703/338184387742924800(0.000000)$
$V_{75}[+9] = 1147/1431468307906560(0.000000)$

$e_{48} = 3288940867/12134707200(0.271036)[35197477704016330752]$
$V_{48}[-7] = 25505321/94928950945382400(0.000000)$
$V_{48}[-6] = 8172767/645775176499200(0.000000)$
$V_{48}[-5] = 1007637241/47464475472691200(0.000000)$
$V_{48}[-4] = 194589824999/163968187996569600(0.000001)$
$V_{48}[-3] = 23433243267407/901825033981132800(0.000026)$
$V_{48}[-2] = 181000587404083/150304172330188800(0.001204)$
$V_{48}[-1] = 174759438479524523/1803650067962265600(0.096892)$
$V_{48}[+1] = 109207543272255949/300608344660377600(0.363288)$
$V_{48}[+2] = 5722102669224553/1803650067962265600(0.003173)$
$V_{48}[+3] = 42223142910659/180365006796226560(0.000234)$
$V_{48}[+4] = 4708918459513/257664295423180800(0.000018)$
$V_{48}[+5] = 18463193263/9394010770636800(0.000002)$
$V_{48}[+6] = 23339473/303644792586240(0.000000)$
$V_{48}[+7] = 1193860483/66801854368972800(0.000000)$
$V_{48}[+8] = 8671/10844456878080(0.000000)$
$V_{48}[+9] = 519961/546560626655232000(0.000000)$

$e_{49} = 10989331/260029440(0.042262)[5602583726551203840]$
$V_{49}[-7] = 131153887/901825033981132800(0.000000)$
$V_{49}[-6] = 8877334111/676368775485849600(0.000000)$
$V_{49}[-5] = 321058535947/676368775485849600(0.000000)$
$V_{49}[-4] = 188187460081/17568020142489600(0.000011)$
$V_{49}[-3] = 149860268269/1294485694709760(0.000116)$
$V_{49}[-2] = 9555417316875511/5410950203886796800(0.001766)$
$V_{49}[-1] = 77997264836317111/541095020388679680(0.144147)$
$V_{49}[+1] = 12026849533501/64724284735488(0.185817)$
$V_{49}[+2] = 5202805469811217/2705475101943398400(0.001923)$
$V_{49}[+3] = 106923208106387/541095020388679680(0.000198)$
$V_{49}[+4] = 24986800232473/1352737550971699200(0.000018)$
$V_{49}[+5] = 2087822266097/2705475101943398400(0.000001)$
$V_{49}[+6] = 87856669279/5410950203886796800(0.000000)$
$V_{49}[+7] = 8813377/644160738557952000(0.000000)$

$e_{50} = 379369/2996224(0.126616)[17127781233131520000]$
$V_{50}[-7] = 2101783/47464475472691200(0.000000)$
$V_{50}[-6] = 164943659/334009271844864000(0.000000)$
$V_{50}[-5] = 465186406151/1803650067962265600(0.000000)$

$V_{50}[-4] = 49542317623/9109343777587200(0.000005)$
$V_{50}[-3] = 6825915940891/100202781553459200(0.000068)$
$V_{50}[-2] = 3811750874427449/1803650067962265600(0.002113)$
$V_{50}[-1] = 29887357972193783/225456258495283200(0.132564)$
$V_{50}[+1] = 6548160578576831/25050695388364800(0.261396)$
$V_{50}[+2] = 708473647746719/901825033981132800(0.000786)$
$V_{50}[+3] = 356805154396393/1803650067962265600(0.000198)$
$V_{50}[+4] = 11564370480493/901825033981132800(0.000013)$
$V_{50}[+5] = 893786131687/225456258495283200(0.000004)$
$V_{50}[+6] = 468039803071/1803650067962265600(0.000000)$
$V_{50}[+7] = 104062907/25050695388364800(0.000000)$
$V_{50}[+8] = 59551/12269728353484800(0.000000)$

$e_{51} = 12998582579/206290022400(0.063011)[8694237351200292864]$
$V_{51}[-8] = 302799841/711967132090368000(0.000000)$
$V_{51}[-7] = 200198636767/1082190040777359360(0.000000)$
$V_{51}[-6] = 4232845423/28478685283614720(0.000000)$
$V_{51}[-5] = 296867080061/110427555181363200(0.000003)$
$V_{51}[-4] = 28279018507781/2705475101943398400(0.000010)$
$V_{51}[-3] = 5285026941479/48312055391846400(0.000109)$
$V_{51}[-2] = 2400296876091767/1352737550971699200(0.001774)$
$V_{51}[-1] = 36173757934527517/238925073937858560(0.151402)$
$V_{51}[+1] = 9847006606815082297/45993076733037772800(0.214098)$
$V_{51}[+2] = 9792585533485231/5410950203886796800(0.001810)$
$V_{51}[+3] = 67216527293609/386496443134771200(0.000174)$
$V_{51}[+4] = 105773214248153/5410950203886796800(0.000020)$
$V_{51}[+5] = 1496553336277/284786852836147200(0.000005)$
$V_{51}[+6] = 1592766194951/5410950203886796800(0.000000)$
$V_{51}[+7] = 10213095919/27606888795340800(0.000000)$
$V_{51}[+8] = 302799841/355983566045184000(0.000000)$

$e_{52} = 4136435441/39437798400(0.104885)[1475572229239681843 2]$
$V_{52}[-7] = 154583/1078738078924800(0.000000)$
$V_{52}[-6] = 8275709/89894839910400(0.000000)$
$V_{52}[-5] = 5082975040331/601216689320755200(0.000008)$
$V_{52}[-4] = 4086365658149/200405563106918400(0.000020)$
$V_{52}[-3] = 146925185217599/450912516990566400(0.000326)$
$V_{52}[-2] = 3225891061904431/901825033981132800(0.003577)$
$V_{52}[-1] = 3629547434141844893/23447450883509452800(0.154795)$
$V_{52}[+1] = 6208463813049931387/23447450883509452800(0.264782)$
$V_{52}[+2] = 2533165683241/2109532243230720(0.001201)$
$V_{52}[+3] = 188653240419367/901825033981132800(0.000209)$
$V_{52}[+4] = 3227651747927/112728129247641600(0.000029)$
$V_{52}[+5] = 3620914111/1937325529497600(0.000002)$
$V_{52}[+6] = 233961671281/1803650067962265600(0.000000)$
$V_{52}[+7] = 3363306073/15030417233018880 0(0.000000)$
$V_{52}[+8] = 35069/28757175828480(0.000000)$

$V_{52}[+9] = 218671/7910745912115200(0.000000)$
$V_{52}[+10] = 10013/47464475472691200(0.000000)$

$e_{53} = 1753484317/36404121600(0.048167)[372961927410354487296]$
$V_{53}[-13] = 1/4831205539184640(0.000000)$
$V_{53}[-12] = 137923/3043659489686323200(0.000000)$
$V_{53}[-11] = 70055/15218297448431616(0.000000)$
$V_{53}[-10] = 24043013971/97397103669962342400(0.000000)$
$V_{53}[-9] = 536992451/81984093998284800(0.000000)$
$V_{53}[-8] = 495270318673/7304782775247175680(0.000000)$
$V_{53}[-7] = 960637729199/24349275917490585600(0.000000)$
$V_{53}[-6] = 214231483919113/292191311009887027200(0.000001)$
$V_{53}[-5] = 56397449178799/13281423227722137600(0.000004)$
$V_{53}[-4] = 3061411267633/216438008155471872(0.000014)$
$V_{53}[-3] = 15076596535394231/97397103669962342400(0.000155)$
$V_{53}[-2] = 250559460613448869/146095655504943513600(0.001715)$
$V_{53}[-1] = 212226881850682202239/1935767435440501555200(0.109634)$
$V_{53}[+1] = 243813332843085582721/1548613948352401244160(0.157440)$
$V_{53}[+2] = 105517436829081911/58438262201977405440(0.001806)$
$V_{53}[+3] = 9720451206859513/48698551834981171200(0.000200)$
$V_{53}[+4] = 332885526845159/16232850611660390400(0.000021)$
$V_{53}[+5] = 2199855412954037/292191311009887027200(0.000008)$
$V_{53}[+6] = 393376392231287/292191311009887027200(0.000001)$
$V_{53}[+7] = 1765246803409/24349275917490585600(0.000000)$
$V_{53}[+8] = 10420346053/76892450265759744(0.000000)$
$V_{53}[+9] = 536992451/40992046999142400(0.000000)$
$V_{53}[+10] = 24043013971/48698551834981171200(0.000000)$
$V_{53}[+11] = 70055/7609148724215808(0.000000)$
$V_{53}[+12] = 137923/1521829744843161600(0.000000)$
$V_{53}[+13] = 1/2415602769592320(0.000000)$

$e_{54} = 32744383961/169885900800(0.192743)[1520584461002805608448]$
$V_{54}[-10] = 519961/2951427383938252800(0.000000)$
$V_{54}[-9] = 254449/10540812085493760(0.000000)$
$V_{54}[-8] = 544051261/491904563989708800(0.000000)$
$V_{54}[-7] = 17186911411/885428215181475840(0.000000)$
$V_{54}[-6] = 267372853813/1545985772539084800(0.000000)$
$V_{54}[-5] = 9682910835401/3607300135924531200(0.000003)$
$V_{54}[-4] = 10292911323827/885428215181475840(0.000012)$
$V_{54}[-3] = 11916594797204821/97397103669962342400(0.000122)$
$V_{54}[-2] = 84872899890985/61839430901563392(0.001372)$
$V_{54}[-1] = 10174542900243355387/97397103669962342400(0.104465)$
$V_{54}[+1] = 7257687613690421939/24349275917490585600(0.298066)$
$V_{54}[+2] = 32905496396048207/32465701223320780800(0.001014)$
$V_{54}[+3] = 294333194742383/4427141075907379200(0.000066)$
$V_{54}[+4] = 630305608616617/32465701223320780800(0.000019)$
$V_{54}[+5] = 378893651143/220855110362726400(0.000002)$

$V_{54}[+6] = 914315912029/2563081675525324800(0.000000)$
$V_{54}[+7] = 1058275765039/32465701223320780800(0.000000)$
$V_{54}[+8] = 10007111909/4058212652915097600(0.000000)$
$V_{54}[+9] = 205655749/2563081675525324800(0.000000)$
$V_{54}[+10] = 4511477/366154525075046400(0.000000)$
$V_{54}[+11] = 2800937/5126163351050649600(0.000000)$
$V_{54}[+12] = 33337/5126163351050649600(0.000000)$

$e_{55} = 943387771/18202060800(0.051829)[416456526393846005760]$
$V_{55}[-10] = 33337/8321693751705600(0.000000)$
$V_{55}[-9] = 83722897/295142738393825280(0.000000)$
$V_{55}[-8] = 1149868206293/146095655504943513600(0.000000)$
$V_{55}[-7] = 1532007576709/9130978469058969600(0.000000)$
$V_{55}[-6] = 50123746848421/97397103669962342400(0.000001)$
$V_{55}[-5] = 9301249107517/1739233994106470400(0.000005)$
$V_{55}[-4] = 967852006786867/58438262201977405440(0.000017)$
$V_{55}[-3] = 6951748079365363/73047827752471756800(0.000095)$
$V_{55}[-2] = 36857804567275229/29219131100988702720(0.001261)$
$V_{55}[-1] = 43645307985941263397/292191311009887027200(0.149372)$
$V_{55}[+1] = 3660220040113447207/18261956938117939200(0.200429)$
$V_{55}[+2] = 442368372384129727/292191311009887027200(0.001514)$
$V_{55}[+3] = 489893102204383/2981543989896806400(0.000164)$
$V_{55}[+4] = 7068099505148281/292191311009887027200(0.000024)$
$V_{55}[+5] = 10499427847969/1014553163228774400(0.000010)$
$V_{55}[+6] = 888556569941/869616997053235200(0.000001)$
$V_{55}[+7] = 98029940871487/292191311009887027200(0.000000)$
$V_{55}[+8] = 4599455666459/292191311009887027200(0.000000)$
$V_{55}[+9] = 83722897/147571369196912640(0.000000)$
$V_{55}[+10] = 33337/4160846875852800(0.000000)$

$e_{56} = 400879067/2316625920(0.173044)[141573769661966684160]$
$V_{56}[-6] = 6785814997/10821900407773593600(0.000000)$
$V_{56}[-5] = 10357492793639/97397103669962342400(0.000000)$
$V_{56}[-4] = 161442370258651/97397103669962342400(0.000002)$
$V_{56}[-3] = 181501819795279/6493140244664156160(0.000028)$
$V_{56}[-2] = 87969285921507127/97397103669962342400(0.000903)$
$V_{56}[-1] = 66389099956276679/512616335105064960(0.129510)$
$V_{56}[+1] = 2903737047336093503/9739710366996234240(0.298134)$
$V_{56}[+2] = 13339984584863687/5410950203886796800(0.002465)$
$V_{56}[+3] = 9234838162290949/24349275917490585600(0.000379)$
$V_{56}[+4] = 88940575640699/1739233994106470400(0.000051)$
$V_{56}[+5] = 10314251483003/1475713691969126400(0.000007)$
$V_{56}[+6] = 75172645647961/48698551834981171200(0.000002)$
$V_{56}[+7] = 9251203375/68348844680675328(0.000000)$
$V_{56}[+8] = 15404457773/2705475101943398400(0.000000)$
$V_{56}[+9] = 4004766797/32465701223320780800(0.000000)$
$V_{56}[+10] = 903919/5072765816143872000(0.000000)$

$V_{56}[+11] = 403/26698767453388800(0.000000)$
$V_{56}[+12] = 41/20291063264575488800(0.000000)$

$e_{57} = 30907469237/509657702400(0.060644)[505006549580537266176]$
$V_{57}[-9] = 9899827187/29219131100988702720\,0(0.000000)$
$V_{57}[-8] = 2078227018577/292191311009887027200(0.000000)$
$V_{57}[-7] = 279694033471/4869855183498117120(0.000000)$
$V_{57}[-6] = 57975124388297/146095655504943513600(0.000000)$
$V_{57}[-5] = 531949181968363/146095655504943513600(0.000004)$
$V_{57}[-4] = 31272327932357/1987695993264537600(0.000016)$
$V_{57}[-3] = 12585903870578537/97397103669962342400(0.000129)$
$V_{57}[-2] = 57593752652897383/32465701223320780800(0.001774)$
$V_{57}[-1] = 43587023408027517991/292191311009887027200(0.149173)$
$V_{57}[+1] = 1530357192955689677/7304782775247175680(0.209501)$
$V_{57}[+2] = 44651341400352107/24349275917490585600(0.001834)$
$V_{57}[+3] = 777758638132831/4058212652915097600(0.000192)$
$V_{57}[+4] = 148197040200331/10821900407773593600(0.000014)$
$V_{57}[+5] = 1954397988395299/292191311009887027200(0.000007)$
$V_{57}[+6] = 182673651588179/292191311009887027200(0.000001)$
$V_{57}[+7] = 1351993586243/121746379587452928 00(0.000000)$
$V_{57}[+8] = 2078227018577/146095655504943513600(0.000000)$
$V_{57}[+9] = 9899827187/146095655504943513600(0.000000)$

$e_{58} = 6267235837/70718054400(0.088623)[750950012179833421824]$
$V_{58}[-9] = 324250979/256308167552532480 0(0.000000)$
$V_{58}[-8] = 23369547097/2563081675525324800(0.000000)$
$V_{58}[-7] = 83894965921/512616335105064960(0.000000)$
$V_{58}[-6] = 287011511657/732309050150092800(0.000000)$
$V_{58}[-5] = 767530296899437/97397103669962342400(0.000008)$
$V_{58}[-4] = 74756283949183/8854282151814758400(0.000008)$
$V_{58}[-3] = 9067008775798753/97397103669962342400(0.000093)$
$V_{58}[-2] = 122681701888511237/97397103669962342400(0.001260)$
$V_{58}[-1] = 189447703516884708733/1412258003214453964800(0.134145)$
$V_{58}[+1] = 207458630795906734361/941505335476302643200(0.220348)$
$V_{58}[+2] = 18651227966840171/8116425305830195200(0.002298)$
$V_{58}[+3] = 2862258848807693/16232850611660390400(0.000176)$
$V_{58}[+4] = 573932994775373/16232850611660390400(0.000035)$
$V_{58}[+5] = 11734282833971/2434927591749058560(0.000005)$
$V_{58}[+6] = 26878790210113/48698551834981171200(0.000001)$
$V_{58}[+7] = 23257660896641/97397103669962342400(0.000000)$
$V_{58}[+8] = 215069764351/19479420733992468480(0.000000)$
$V_{58}[+9] = 230825334239/48698551834981171200(0.000000)$
$V_{58}[+10] = 5755065131/24349275917490585600(0.000000)$
$V_{58}[+11] = 68623789/24349275917490585600(0.000000)$

$e_{59} = 163742997419/3057946214400(0.053547)[2769322008374912876544 0]$
$V_{59}[-15] = 41/6763687754858496000(0.000000)$
$V_{59}[-14] = 61/84546096935731200(0.000000)$
$V_{59}[-13] = 1319/48312055391846400(0.000000)$
$V_{59}[-12] = 2483197/6763687754858496000(0.000000)$
$V_{59}[-11] = 4760227907/14609565550494351360 00(0.000000)$
$V_{59}[-10] = 36261195821/29219131100988702720 0(0.000000)$
$V_{59}[-9] = 99251424791/9130978469058969600 0(0.000000)$
$V_{59}[-8] = 145306469550229/58438262220197740544000(0.000000)$
$V_{59}[-7] = 96842410787587/92270940318911692800 0(0.000000)$
$V_{59}[-6] = 908924421985849/834832317171105792000(0.000001)$
$V_{59}[-5] = 15866293406797/2385235191917445120(0.000007)$
$V_{59}[-4] = 7538369215801399/273929354071769088000(0.000028)$
$V_{59}[-3] = 469300376053615651/1593770787326656512000(0.000294)$
$V_{59}[-2] = 30998833972906813663/17531478660593221632000(0.001768)$
$V_{59}[-1] = 10133633382323958190269 1/1034357240975000076288000(0.097970)$
$V_{59}[+1] = 157314414346394846410 49/1034357240975000076288 00(0.152089)$
$V_{59}[+2] = 503906908616282993/318754157465331302400(0.001581)$
$V_{59}[+3] = 2029679255343050999/8765739330296610816000(0.000232)$
$V_{59}[+4] = 17982332246708579/626124237878329344000(0.000029)$
$V_{59}[+5] = 453773605688981/106251385821777100800(0.000004)$
$V_{59}[+6] = 147689548546613/166966463434221158400(0.000001)$
$V_{59}[+7] = 1221525888512777/17531478660593221632000(0.000000)$
$V_{59}[+8] = 116187185196623/58438262220197740544000(0.000000)$
$V_{59}[+9] = 5560595677129/58438262220197740544000(0.000000)$
$V_{59}[+10] = 147844102799/1460956555049435136000(0.000000)$
$V_{59}[+11] = 1769351693/4174161585855528960 00(0.000000)$
$V_{59}[+12] = 2483197/3381843877429248000(0.000000)$
$V_{59}[+13] = 1319/24156027695923200(0.000000)$
$V_{59}[+14] = 61/42273048467865600(0.000000)$
$V_{59}[+15] = 41/3381843877429248000(0.000000)$

$e_{60} = 67410047209/254828851200(0.264531)[1391284148465562550272 00]$
$V_{60}[-11] = 48572009/3075698010630389760 00(0.000000)$
$V_{60}[-10] = 7564923041/1025232670210129920 00(0.000000)$
$V_{60}[-9] = 241946098633/6151396021260779520 0(0.000000)$
$V_{60}[-8] = 82710948197627/5843826220197740544000(0.000000)$
$V_{60}[-7] = 618201928419193/1460956555049435136000(0.000000)$
$V_{60}[-6] = 3617639494318727/1460956555049435136000(0.000002)$
$V_{60}[-5] = 25317575928648679/1947942073399246848000(0.000013)$
$V_{60}[-4] = 56398540189930703/531256929108885504000(0.000106)$
$V_{60}[-3] = 93326831348910953/584382622019774054400(0.000160)$
$V_{60}[-2] = 2254270194640917743/834832317171105792000(0.002700)$
$V_{60}[-1] = 63503355003238436449/649314024466415616000(0.097801)$
$V_{60}[+1] = 2119164534676506715771/5843826220197740544000(0.362633)$
$V_{60}[+2] = 2258443180589756309/834832317171105792000(0.002705)$
$V_{60}[+3] = 105935771427821093/531256929108885504000(0.000199)$
$V_{60}[+4] = 540048225702181/39753919865290752000(0.000014)$
$V_{60}[+5] = 3885941279688631/973971036699623424000(0.000004)$
$V_{60}[+6] = 7952143409509/26088509911597056000(0.000000)$
$V_{60}[+7] = 68408181635527/973971036699623424000(0.000000)$
$V_{60}[+8] = 119925986148293/5843826220197740544000(0.000000)$
$V_{60}[+9] = 5490125991767/5843826220197740544000(0.000000)$
$V_{60}[+10] = 244847740997/3075698010630389760000(0.000000)$
$V_{60}[+11] = 14716614571/973971036699623424000(0.000000)$
$V_{60}[+12] = 2237256377/243492759174905856000(0.000000)$
$V_{60}[+13] = 30677699/177085643036295168000(0.000000)$

$e_{61} = 19902302711/764486553600(0.026034)[863065094810581724037120]$
$V_{61}[-13] = 33337/19223112566439936000(0.000000)$
$V_{61}[-12] = 26683982221/18115861282612995686400(0.000000)$
$V_{61}[-11] = 4775116095173/30193102137688326144000(0.000000)$
$V_{61}[-10] = 69756035473169/15096551068844163072000(0.000000)$
$V_{61}[-9] = 312279556434773/4528965320653248921600(0.000000)$
$V_{61}[-8] = 72308884322139191/36231722565225991372800(0.000000)$
$V_{61}[-7] = 34398025829829313/1207724085507533045760(0.000000)$
$V_{61}[-6] = 457539002368786193/54347583847838987059200(0.000001)$
$V_{61}[-5] = 2575014867714166963/181158612826129956864000(0.000014)$
$V_{61}[-4] = 3066031235443749497/155278810993825677312000(0.000020)$
$V_{61}[-3] = 15639813937655487379/67934479809798733824000(0.000230)$
$V_{61}[-2] = 1423557203480883778099/1086951676956779741184000(0.001310)$
$V_{61}[-1] = 2061939117042950979905/13958747851444960886784(0.147717)$
$V_{61}[+1] = 11468365862330930545275961/66304052294363564212224000(0.172966)$
$V_{61}[+2] = 44062736926197420397/24703447203563175936000(0.001784)$
$V_{61}[+3] = 25978409273977745917/135868959619597467648000(0.000191)$
$V_{61}[+4] = 19021961155258063699/1086951676956779741184000(0.000018)$
$V_{61}[+5] = 89329015884779113/12077240855075330457600(0.000007)$
$V_{61}[+6] = 2382219017953771/5200725727065931776000(0.000000)$
$V_{61}[+7] = 4359398737953311/25879801832304279552000(0.000000)$
$V_{61}[+8] = 185240586402109/1848557273736019968000(0.000000)$
$V_{61}[+9] = 312280878311483/90579306413064978432000(0.000000)$
$V_{61}[+10] = 69756035473169/30193102137688326144000(0.000000)$
$V_{61}[+11] = 4775116095173/60386204275376652288000(0.000000)$
$V_{61}[+12] = 26683982221/36231722565225991372800(0.000000)$
$V_{61}[+13] = 33337/38446225132879872000(0.000000)$

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